A PUBLICATION FOR THE MIDDLE SCHOOLS OF THE REPUBLIC OF SENEGAL
APPROVED BY THE MINISTRY OF EDUCATION

Mathematics

Quantitative Reasoning

Grade 7 - 8

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Theme I: Rational Numbers

Lesson 1: The Properties of Rational Numbers

Do You Know?

A **set** is a well – defined collection of objects. An object which belongs to the collection is called an **element** or **member** of the set. Example: A = {x, y, z}; z is an element of the set A. In mathematics, we frequently make use of the following important sets of numbers.

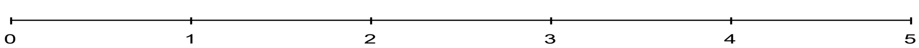
The set of **natural numbers** (or **counting numbers**) is

{1, 2, 3,…}

We can represent the natural numbers on a **number line** with equally spaced dots beginning at 1 and continuing to the right forever.

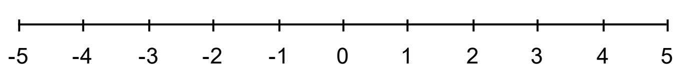
The set of **whole numbers** is the same as the set of natural numbers except it includes zero. Thus, its members are {0, 1, 2, 3,…}

We can represent the whole numbers on a number line with equally spaced dots beginning at 0 and continuing to the right forever.



The set of **integers** includes the whole numbers and their negatives. Thus, its members are {…, -3, -2, -1, 0, 1, 2, 3,…}

On the number line, the integers extend forever both to the left and to the right.



The set of **rational numbers** includes the integers and the fractions that can be made by dividing one integer by another, as long as we do not divide by zero. (The word rational refers to a ***ratio*** of integers.) In other words, rational numbers can be expressed in the form

$\frac{x}{y}$ where x and y are integers and y $\neq$ 0

(Recall that the symbol $\neq$ means "is not equal to.")

The Properties of Rational Numbers

The Rational Number System

This lesson explores the properties of the rational numbers. Many properties will be familiar, since the integers have corresponding properties. However, we will also discover some important new properties of rational numbers which have no counterpart in the integers. This lesson also gives techniques for estimation and computation. It also gives examples of the application of rational numbers being used as solutions of practical problems that involves quantitative reasoning.

Properties of Addition and Subtraction

Negative or Additive Inverse

Let $\frac{a}{b}$ be a rational number. Its **negative,** or **additive inverse,** written - $\frac{a}{b}$, is the rational number $\frac{-a}{b}$. Example $\frac{5}{7}$ additive inverse is $\frac{-5}{7}$

Properties of Addition of Rational Numbers

Let $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{e}{f}$ be rational numbers. The following properties hold.

Closure Property $\frac{a}{b} + \frac{c}{d}$ The sum of two rational numbers. Is a rational number.

Commutative Property $\frac{a}{b} + \frac{c}{d} = \frac{c}{d} + \frac{a}{b}$

Associative Property $\left(\frac{a}{b} + \frac{c}{d}\right) + \frac{e}{f} = \frac{a}{b} + \left(\frac{c}{d} + \frac{e}{f}\right)$

Zero (0) is the Additive Identity $\frac{a}{b} + 0 = \frac{a}{b}$

Existence of Additive Inverses $\frac{a}{b} + \left(-\frac{a}{b}\right) = 0$, where $-\frac{a}{b} = \frac{-a}{b}$

The Properties of Rational Numbers

Formulas for Subtraction of rational Numbers

Let $\frac{a}{b}$ and $\frac{c}{d}$ be rational numbers. Then $\frac{a}{b} - \frac{c}{d} = \frac{a}{b} + \left(-\frac{c}{d}\right) = \frac{ad-bc}{bd}$.

Properties of Multiplication of Rational Numbers

Let $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{e}{f}$ be rational numbers. The following properties hold.

Closure Property $\frac{a}{b} \cdot \frac{c}{d}$ the product of two rational number is a rational number.

Commutative Property $\frac{a}{b} \cdot \frac{c}{d} = \frac{c}{d} \cdot \frac{a}{b}$

Associative Property $\left(\frac{a}{b} \cdot \frac{c}{d}\right) \cdot \frac{e}{f} = \frac{a}{b} \cdot \left(\frac{c}{d} \cdot \frac{e}{f}\right)$

Distributive Property of Multiplication over Addition and Subtraction

$\frac{a}{b} \cdot \left(\frac{c}{d} + \frac{e}{f}\right) = \frac{a}{b} \cdot \frac{c}{d} + \frac{a}{b} \cdot \frac{e}{f}$ and $\frac{a}{b} \cdot \left(\frac{c}{d} - \frac{e}{f}\right) = \frac{a}{b} \cdot \frac{c}{d} - \frac{a}{b} \cdot \frac{e}{f}$

Multiplication by Zero $0 \cdot \frac{a}{b} = 0$

One (1) is the Multiplicative Identity $1 \cdot \frac{a}{b} = \frac{a}{b}$

Existence of Multiplicative Inverse If $\frac{a}{b} \neq 0$, then there is a unique rational numbers, namely $\frac{b}{a}$, for which $\frac{a}{b} \cdot \frac{b}{a} = 1$.

Properties of the Order Relation on the Rational Numbers

Let $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{e}{f}$ be rational numbers.

The Properties of Rational Numbers

Transitive Property

If $\frac{a}{b} < \frac{c}{d}$ and $\frac{c}{d} < \frac{e}{f}$, then $\frac{a}{b} < \frac{e}{f}$.

Addition Property

If $\frac{a}{b} < \frac{c}{d}$, then $\frac{a}{b} + \frac{e}{f} < \frac{c}{d} + \frac{e}{f}$.

Multiplication Property

If $\frac{a}{b} < \frac{c}{d}$ and $\frac{e}{f} > 0$, then $\frac{a}{b} \cdot \frac{e}{f} < \frac{c}{d} \cdot \frac{e}{f}$.

If $\frac{a}{b} < \frac{c}{d}$ and $\frac{e}{f} < 0$, then $\frac{a}{b} \cdot \frac{e}{f} > \frac{c}{d} \cdot \frac{e}{f}$.

Trichotomy Property Only one of the following is true:

$\frac{a}{b} < \frac{c}{d}$ **,** $\frac{a}{b} = \frac{c}{d}$ **, or** $\frac{a}{b} > \frac{c}{d}$ **.**

The Density Property of Rational Numbers

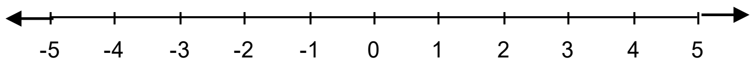
Let $\frac{a}{b}$ and $\frac{c}{d}$ be any two rational numbers, with $\frac{a}{b} < \frac{c}{d}$. Then there is a rational number $\frac{e}{f}$ between $\frac{a}{b}$ and $\frac{c}{d}$; that is, $\frac{a}{b} < \frac{e}{f} < \frac{c}{d}$.

The Properties of Rational Numbers

Activity 1

1. Label these points on the number line

a) $\frac{3}{4}$ **b)** $-\frac{7}{8}$ **c)** $\frac{12}{18}$ **d)** -2 **e)** $2\frac{3}{8}$



2. List the additive inverse of the fractions in Problem 1, for

a) **b)** **c)** **d)** **e)**

3. List the multiplicative inverses of the fractions in Problem 1, for

a) **b)** **c)** **d)** **e)**

4. Order the fractions in Problem 1 from the smallest to the largest.

5. Illustrate the Closure Property of Addition using the fractions in **b)** and **c)** **(in Problem 1)**; give the final results as a single fraction.

6. Illustrate the Closure Property of Multiplication using the fractions in **a)** and **c)** **(in Problem 1)**; give the final results as a single fraction.

7. Illustrate the Commutative Property of Addition using the fractions in **b)** and **c)** **(in Problem 1)**; give the final results as a single fraction on each side of the equality sign.

8. Illustrate the Commutative Property of Multiplication using fractions in **a)** and **c)** **(in Problem 1)**; give the final results as a single fraction on each side of the equality sign.

9. Illustrate the Associative Property of Addition using the fractions **a), b)** and **c)** **(in Problem 1)**; give the details in each step and give the results as a single fraction on each side of the equality sign.

10. Illustrate the Distributive Property of Multiplication Over Addition using the fractions **a), b)** and **c) (in Problem 1)**; give the details in each step and give the final results as a single fraction on each side of the equality sign.

The Properties of Rational Numbers

Activity 2

1. **Discuss and illustrate the differences between natural numbers, whole numbers, integers and rational numbers (fractions).**

2. **Discuss and illustrate the density of rational numbers; use the actual fractions (numbers) and give several examples.**

3. **Discuss why you think that the additive identity, zero (0), and the multiplicative identity, one (1) is important or is not important; illustrate several uses of both zero (0) and one (1).**

4. **Discuss and illustrate the transitive properties of rational numbers; give several examples.**

5. **Discuss and illustrate the order of rational numbers when the same fraction is added or multiplied to two fractions, one (1) fraction being less or greater than the others.**

Theme I: Rational Numbers

Lesson 2: Rational Numbers and Arithmetic Operations

Do You Know?

Operations of Rationals

1. **The Basic Concepts of Fractions and Rational Numbers**

(a) A fraction is an ordered pair of integers a and b, $b \neq 0$, written $\frac{a}{b}$ or a/b.

(b) Two fractions that express the same quantity, or correspond to the same point on a number line, are called equivalent fractions. In particular $\frac{a}{b} = \frac{a \cdot n}{b \cdot n}$ for all integers n, $n \neq 0$ and $\frac{a}{b} = \frac{a \div d}{b \div d}$ if d divides a and b (<u>the fundamental law of fractions</u>), and $\frac{a}{b} = \frac{c}{d}$ if, and only if, $ad = bc$.

(c) Any fraction is equivalent to a fraction in simplest form. Two or more fractions can always be replaced by equivalent fractions with a common denominator.

(d) A rational number is a number represented by a common fraction $\frac{a}{b}$. The same rational number can also be represented by any fraction equivalent to $\frac{a}{b}$.

(e) If two rational numbers are represented by $\frac{a}{b}$ and $\frac{c}{d}$, with $b > 0$ and $d > 0$, then $\frac{a}{b} < \frac{c}{d}$ if, and only if, $ad < bc$.

Rational Numbers and Arithmetic Operations

2. The Arithmetic of Rational Numbers

(a) The sum of two rational numbers represented by fractions $\frac{a}{b}$ and $\frac{c}{b}$ with a common denominator is defined by $\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}$. From this it follows that $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$.

(b) Subtraction is defined by the missing-addend approach: $\frac{a}{b} - \frac{c}{d} = \frac{e}{f}$ if, and only if, $\frac{a}{b} = \frac{c}{d} + \frac{e}{f}$. The subtraction formula $\frac{a}{b} - \frac{c}{d} = \frac{ad-bc}{bd}$ follows from the definition of subtraction.

(c) Multiplication is defined by $\frac{a}{b} \bullet \frac{c}{d} = \frac{ac}{bd}$.

(d) This definition is motivated by extending the rectangular array model of multiplication.

(e) Division is defined by the missing-factor approach: $\frac{a}{b} \div \frac{c}{d} = \frac{e}{f}$ if, and only if $\frac{a}{b} = \frac{c}{d} \bullet \frac{e}{f}$.

(f) A nonzero rational number, $\frac{a}{b}$, has a unique multiplicative inverse, the reciprocal $\frac{b}{a}$, which when multiplied by $\frac{a}{b}$ gives the product 1. That is, $\frac{a}{b} \bullet \frac{b}{a} = 1$.

3. The rational Number System

(a) The rational numbers are closed under addition. Addition is commutative, associative, and zero is the additive identity.

(b) Each rational number, $\frac{a}{b}$, has a unique negative, $-\frac{a}{b}$, given by $\frac{-a}{b}$, which is the additive inverse of $\frac{a}{b}$. Subtraction is equivalent to adding the negative , so $\frac{a}{b} - \frac{b}{d} = \frac{a}{b} + (-\frac{b}{d})$.

(c) Multiplication is closed, commutative, associative, one is the multiplicative identity, and multiplication distributes over addition and subtraction.

(d) Each nonzero rational number $\frac{a}{b}$ has a unique multiplicative inverse given by the reciprocal $\frac{b}{a}$. Division is equivalent to multiplication by the multiplicative inverse of the divisor, so $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \bullet \frac{d}{c}$.

(e) The rational numbers have the density property; that is, there is a rational number between any two rational numbers.

(f) Computational skills – estimations, mental arithmetic, paper and pencil and electronic calculations – are as useful and necessary for work with the rational numbers as for any other number system.

Common Fractions

Common fractions have the form $\frac{a}{b}$ or *a*/*b*, where *a* and *b* can be any number as long as *b* is not zero. The number on top is called the numerator and the number on the bottom is called the denominator. A fraction represents division:

numerator –
denominator - $\frac{a}{b}$ means $a \div b$

We can write an integer as a fraction with a denominator of 1. For example:

$3 = \frac{3}{1}$ or $-4 = \frac{-4}{1}$

Rational Numbers and Arithmetic Operations

Simplest Form

A fraction $\frac{a}{b}$ is in simplest form if there is not a number *c* that divides both *a* and *b* except 1. $\frac{5}{9}$ is in simplest form. $\frac{6}{9}$ is not because $\frac{6}{9} = \frac{2 \cdot 3}{3 \cdot 3} = \frac{2}{3}$ (which is in simplest form).

Example

$$
\begin{aligned}
\text{Solve } 3\frac{1}{5} + 5\,\frac{3}{4} &= \left(3 + \frac{1}{5}\right) + \left(5 + \frac{3}{4}\right) \\
&= \left(\frac{5}{5} \cdot \frac{3}{1} + \frac{1}{5}\right) + \left(\frac{4}{4} \cdot \frac{5}{1} + \frac{3}{4}\right) \\
&= \left(\frac{15}{5} + \frac{1}{5}\right) + \left(\frac{20}{1} + \frac{3}{4}\right) \\
&= \left(\frac{16}{5} + \frac{23}{4}\right) \\
&= \frac{\cancel{20}}{20} \cdot \frac{16}{\cancel{5}} + \frac{\cancel{20}}{20} \cdot \frac{23}{\cancel{4}} \\
&= \frac{64}{20} + \frac{115}{20} \\
&= \frac{179}{20}
\end{aligned}
$$

Rational Numbers and Arithmetic Operations

Adding and Subtracting Fractions

If two fractions have a common (same) denominator, we can add or subtract them by adding or subtracting their numerators. For example:

$$\frac{1}{5}+\frac{2}{5}=\frac{1+2}{5}=\frac{3}{5} \qquad \text{or} \qquad \frac{7}{9}-\frac{2}{9}=\frac{7-2}{9}=\frac{5}{9}$$

Otherwise, we must write the fractions with a common denominator before adding or subtracting. For example, we can add $\frac{1}{2}+\frac{1}{3}$ by writing them with a common denominator of 6 as $\frac{3}{6}$ and $\frac{2}{6}$, respectively:

$$\frac{1}{2}+\frac{1}{3}=\frac{3}{6}+\frac{2}{6}=\frac{3+2}{6}=\frac{5}{6}$$

Multiplying Fractions

To multiply fractions, we multiply the numerators and denominators separately. For example:

$$\frac{1}{3}\times\frac{2}{5}=\frac{1\times 2}{3\times 5}=\frac{2}{15}$$

Sometimes we can simplify fractions at the same time we multiply them by *canceling* terms that occur in both the numerator and the denominator. For example:

$$\frac{3}{4}\times\frac{5}{3}=\frac{\cancel{3}\times 5}{4\times\cancel{3}}=\frac{5}{4}$$

Reciprocals and Division

Two numbers are **reciprocals** if their product is 1. For example:

$$2 \text{ and } \frac{1}{2} \text{ are reciprocals because } 2\times\frac{1}{2}=1$$

$$\frac{4}{3} \text{ and } \frac{3}{4} \text{ are reciprocals because } \frac{4}{3}\times\frac{3}{4}=1$$

We find the reciprocal of any fraction *y* inverting it (interchanging the numerator and the denominator). For an integer like 2, we think of it as $\frac{2}{1}$ to find that its reciprocal is $\frac{1}{2}$.

Rational Numbers and Arithmetic Operations

Activity 1

***Given the fractions a)** $\frac{3}{8}$ **b)** $5\frac{1}{4}$ **c)** $\frac{35}{4}$ **d)** $\frac{-4}{27}$ **e)** $2\frac{5}{12}$ **f)** $\frac{123}{369}$ **g)** $\frac{1}{3}$

1. Which of the fractions in (*) is in simplest form?

2. Give at least one equivalent fraction in the form $\frac{a}{b}$ for each of the fractions in (*).

3. Add the fractions $\frac{3}{8} + 5\frac{1}{4} + \frac{35}{4}$.

4. Multiply the fractions $\frac{3}{8} \times 5\frac{1}{4} \times \frac{35}{4}$.

5. Add the fractions $(5\frac{1}{4} + 2\frac{5}{12}) \cdot \frac{-4}{27}$.

6. Divide $\frac{123}{369} \div \frac{1}{3}$, mentally.

7. Divide $(\frac{-4}{27} + \frac{35}{4}) \div \frac{-4}{27}$.

8. Calculate $5\frac{1}{4} + 2\frac{5}{12}$ and $5\frac{1}{4} - 2\frac{5}{12}$.

9. Calculate $\frac{35}{4} \times 2\frac{5}{12}$ and $\frac{35}{4} \div 2\frac{5}{12}$.

10. Subtract $(\frac{35}{4} - 5\frac{1}{4}) \div 2\frac{5}{12}$.

Rational Numbers and Arithmetic Operations

Activity 2

What is the best approximate answer listed for each of these problems?

1. $2\frac{1}{48}+3\frac{1}{99}+6\frac{13}{25}$ is approximately

a) 11 **b)** $11\frac{1}{2}$ **c)** 12 **d)** $12\frac{1}{4}$ **e)** none of these

2. $8\cdot\left(2\frac{1}{2}+3\frac{7}{15}\right)$ is approximately

a) 40 **b)** 44 **c)** 48 **d)** 56 **e)** none of these

3. $11\frac{9}{10}\div\frac{21}{40}$ is approximately

a) 20 **b)** 23 **c)** 26 **d)** 30 **e)** none of these

4. Describe how the following calculations can be performed efficiently with "mental math" (no writing).

a) $\frac{19}{111}\cdot\left(\frac{2}{3}+\frac{-4}{6}\right)$ **b)** $\frac{5}{6}\cdot\frac{36}{15}$

5. Describe how the following calculations can be performed efficiently with "mental math" (no writing).

a) $\frac{5}{8}\cdot\left(\frac{9}{5}-\frac{1}{5}\right)$ **b)** $\frac{2}{3}\cdot\frac{3}{4}\cdot\frac{4}{5}\cdot\frac{5}{6}$

Theme I: Rational Numbers

Lesson 3: Counting With Factorials, Permutations, and Combinations

Do You Know?

Factorials

Products of the form $4 \times 3 \times 2 \times 1$ come up so frequently in counting problems that they have a special name. Whenever a positive integer n is multiplied by all the preceding positive integers, the result is called ***n* factorial** and is denoted $n!$ (the exclamation mark is read as "factorial"). For example:

$1! = 1$
$2! = 2 \times 1 = 2$
$3! = 3 \times 2 \times 1 = 6$
$4! = 4 \times 3 \times 2 \times 1 = 24$
$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

In general,

$$n! = n \times (n - 1) \times (n - 2) \times \cdots \times 2 \times 1$$

Note that $n!$ grows rapidly with n. For example $20! \approx 2.4 \times 10^{18}$, $40! \approx 8.2 \times 10^{47}$, and $60! \approx 8.3 \times 10^{81}$. In fact, factorials quickly become so large that many calculators cannot handle them above $n = 69$ (if the calculator limit is 10^{100}).
Note also that, by definition, $0! = 1$.

Example Calculate each of the following *without* using the factorial key on you calculator.

a. $\frac{6!}{4!}$

Solution

a. We can write out the entire calculation, but it is easier if we simplify it by canceling like terms in the numerator and denominator:

$$\frac{6!}{4!} = \frac{6 \times 5 \times \cancel{4} \times \cancel{3} \times \cancel{2} \times 1}{\cancel{4} \times \cancel{3} \times \cancel{2} \times 1} = \frac{6 \times 5 \times 4!}{4!}$$

$$= 6 \times 5 = 30$$

Counting With Factorials, Permutations, and Combinations

Permutations

Mathematically, we are dealing with permutations whenever all selections come from a single group of items, no item may be selected more than once and the *order of arrangement matter* (for example, ABCD is considered different from DCBA). The total number of permutations possible with a group of N items is n!, where n! = n * (n-1) *…*2*1 is read "n factorial."

The Permutations Formula

Permutations have their own special notation: We read ${}_{10}P_4$ as "the number of permutations of ten items selected four at a time.' Using this compact notation, we have

$${}_{10}P_4 = \frac{10!}{(10-4)!} = 5040$$

Generalizing, we have a formula for calculating the number of permutations.

If we make *r* selections from a group of *n* items, the number of permutations is

$${}_{n}\mathrm{P}_{r} = \frac{n!}{(n-r)!} = \underbrace{n \times (n-1) \times (n-2) \times \cdots \times (n-r+1)}_{r factors here}$$

where ${}_{n}\mathrm{P}_{r}$ is read as "the number of permutations of *n* items taken *r* at a time." That is, there are 60 possible permutations when we choose three people from a group of five.

However, because order matters for permutations but does not matter for committees, the number of permutations is an **overcount** of the actual number of different committees. More specifically, any three-person committee can be listed in 3! = 3 × 2 × 1 = 6 different orders. For example, the committee consisting of Zeke, Yolanda, and Wendy has six (6) different permutations:

ZYW ZWY YZW YWZ WZY WYZ

Because each three-person committee is counted 3! = 6 times by the permutations formula, this formula gives us six times the actual number of committees. Thus, we must divide the number of permutations by 3! to find the number of committees:

$$\frac{{}_{5}P_{3}}{3!} = \frac{60}{3!} = \frac{60}{3 \times 2 \times 1} = 10$$

This is the same result we obtained by listing the three-person committees. The permutations part of the equations is ${}_{n}\mathrm{P}_{r}$, where $n = 5$ and r = 3. We then divided this term by $r! = 3!$ to correct for overcounting. Thus, the general formula for combinations is the permutations formula divided the $r!$.

Counting With Factorials, Permutations, and Combinations

Combinations

Combinations occur whenever all selections come from a single group of items, no item may be selected more than once, and the order of arrangement does not matter (for example, ABCD is considered to be the same as DCBA). If we make r selections from a group of n items, the number of possible combinations is

$${}_nC_r = \frac{{}_nP_r}{r!} = \frac{n!}{(n-r)! \times \; r!}$$

where ${}_nC_r$ is read as "the number of combinations of *n* items taken *r* at a time."

Example Suppose that you select three (3) different flavors of ice cream in a shop that carries twelve (12) flavors. How many flavor combinations are possible?

Solution We are looking for the number of combinations of $n = 12$ flavors selected $r = 3$ at a time. From the combinations formula, the number of flavor combinations is

$${}_{12}C_3 = \frac{12!}{(12-3)! \times \; 3!} = \frac{12!}{9! \times 3!} = \frac{12\times 11\times 10\times 9!}{9! \times 3!} = \frac{12\times 11\times 10}{3 \times 2 \times 1} = \frac{1320}{6} = 220$$

There are 220 different three-flavor combinations possible from the 12 flavors.

Counting With Factorials, Permutations, and Combinations

Activity 1

1. What number is 6!

2. What is $\frac{25!}{22!}$

3. What is $\frac{200!}{199!}$

4. List all permutations of $\{w, x, y, z\}$

Let s = {1, 2, 3, 4, 5}

5. List all the 3-permutations of *s*.

6. List all the 3-combinations of *s*.

7. Find the value of

a) P(6, 3) **b)** P(8, 5)

8. Find the value of

a) C(8, 4) **b)** C(12, 6)

9. What relationship exists, if any between P(4, 3) and C(4, 3)?
(Hint: first calculate each)

10. Which of the following are the same?

a) 0! **b)** 1! **c)** $\frac{6!}{6!}$ **d)** P(6, 6) **e)** C(6, 6)

Counting With Factorials, Permutations, and Combinations

Activity 2

1. A middle school principal needs to schedule six different classes-Algebra, English, History, Spanish, Science, and Gym- in six different time periods. How many different class schedules are possible?

2. A city has 12 candidates running for three leadership positions. The top vote-getter will become the mayor, the second vote-getter will become the deputy mayor, and the third vote-getter will become the treasurer. How many outcomes are possible for the three leadership positions?

3. Suppose that there are eight runners in a race. The winner receives a gold medal, the second-place finisher receives a silver medal, and the third-place finisher receives a bronze medal. How many different ways are there to award these medals, if all possible outcomes of the race can occur?

4. Let $s = \{a, b, c, d, e, f\}$

How many subsets of *s* exist that contain *(Hint: combinations)*

a) no element?
b) exactly one (1) element
c) exactly two (2) elements
d) exactly three (3) elements
e) exactly four (4) elements
f) exactly five (5) elements
g) exactly six (6) elements
h) at least four (4) elements

5. List and discuss two major difference between permutations and combinations.

Theme I: Rational Numbers

Lesson 4: Sequences and Summations

Do You Know?

Sequences are used to represent ordered lists of elements. Sequences are used in discrete mathematics in many ways. Sequences can be used to represent solutions to certain counting problems. This lesson will present the notation used to represent sequences and sums of terms of sequences. When the elements of an infinite set can be listed, in some systematic way, the set is called **countable**. We will conclude this lesson with a discussion of both countable and uncountable sets.

A **sequence** is a mathematical structure used to represent an ordered list. A sequence is a list of items that can be arranged in a 1-1 correspondence with a subset of the positive integers or whole numbers (usually either the set {0, 1, 2, …} or the set {1, 2, 3, ….}) to a set S. We use the notation a_n to denote the nth term of the sequence. We call a_n a *term* of the sequence.

We use the notation $\{a_n\}$ to describe the entire sequence. Observe that a_n represents an individual term of the sequence $\{a_n\}$. Also, observe that the notation $\{a_n\}$for a sequence is not the usual notation for a set. However, the context in which we use this notation will always make it clear when we are discussing sets and when we are discussing sequences. We describe sequences by listing the terms of the sequence in order of increasing subscripts.

EXAMPLE 1 Consider the sequence $= \{a_n\}$, where

$$a_n = 1/n.$$

The list of the terms of this sequence, beginning with a_1, namely,

$$a_1, a_2, a_3, a_4, \ldots.,$$

starts with

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots.,$$

EXAMPLE 2 Sequences of the form

$$a_1, a_2, \ldots, a_n,$$

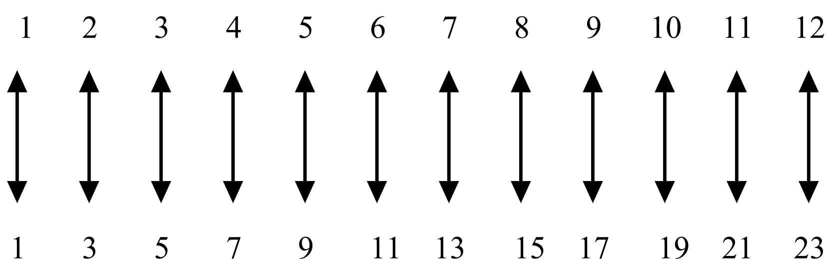
are often used in mathematics. These finite sequences are also called **strings**. This string is also denoted by $a_1\ a_2 \ldots a_n$. The **length** of the string S is the number of terms in this string. The empty string is the string that has no terms. The **empty string** has length zero.

Sequences and Summations

EXAMPLE 3 The string *abcd* is a string of length four.

Special Integers Sequences

A common problem in quantitative mathematics is finding a formula or a general rule for constructing the terms of a sequence. Even though the initial terms of a sequence do not determine the entire sequence, they can usually help us find how to identify the n^{th} term of the sequence or how to identify all terms of the sequence.



A One-to-One Correspondence Between N and the Set of Odd Positive Integers.

An infinite set is countable if and only if it is possible to list the elements of the set in a one-to-one correspondence from the set of natural numbers to a set S. This can be expressed in terms of a sequence $a_1, a_2, \ldots, a_n, \ldots$. For instance, the set of odd integers can be listed in a sequence $a_1, a_2, \ldots, a_n, \ldots$., where $a_n = 2n - 1$ (see the graph/figure on this page).

Cardinality

We define the **cardinality** of a finite set to be the number of elements in the set. It is possible to extend the concept of cardinality to all sets, both finite and infinite, with the following definition.

Definition The sets *A* and *B* have the same **cardinality** if and only if there is a one-to-one correspondence from *A* to *B*.

We will now classify infinite sets into two groups, those with the same cardinality as the set of natural numbers and those with different cardinalty.

Definition A set that is either finite of has the same cardinality as the set of natural numbers is called **countable**. A set that is not countable is called **uncountable**.

We now give examples of countable and uncountable sets.

Example We have shown that the set of odd positive integers is a countable set. The set of all fractions between 0 and 1 can be shown to be uncountable.

Sequences and Summations

Summation

Here we introduce the summation symbol $\sum$ (the Greek letter sigma). Consider a sequence a_1, a_2, a_3….. Then the sums

$$a_1 + a_2 \ldots + a_n \qquad \textit{and} \qquad a_m + a_{m+1} + \ldots + a_n$$

will be denoted, respectively by

$$\sum_{j=1}^{n} a_j \qquad \text{and} \qquad \sum_{j=m}^{n} a_j$$

*The letter j is called a dummy index or a dummy variable.

Example What is the value of $\sum_{j=1}^{5} j^2$

Solution We have $\sum_{j=1}^{5} j^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2$

$= 1 + 4 + 9 + 16 + 25$

$= 55$

Some special properties of sequences and summations

- **Arithmetic Sequence** Any sequence in which any two (2) consecutive terms have the same common difference (d) is called an **arithmetic sequence.**
  Example In the sequence {1, 4, 7, 10, 13, …} every two terms have a common difference, $d = 3$.

- **Geometric Sequence** Any sequence where any two (2) consecutive terms have a common ratio (r) is called a **geometric sequence**.
  Example In the sequence {3, 6, 12, 24, 48, …} note that

  $$\frac{6}{3} = 2, \quad \frac{12}{6} = 2, \quad \frac{24}{12} = 2, \quad \frac{48}{24} = 2$$

- **Recursive Sequences** When a sequence is defined by an initial condition and a **recurrence relation**, we call this type of sequence a **recursive sequence**.
  Example a) $a_1 = 1$
  b) $a_n = a_{n-1} + 3$, for $n \geq 2$
  The sequence is {1, 4, 7, 10, 13, 16, …}

- **Double Summations** Some summations involve two or more summation symbols.
  Example

  $$\sum_{i=1}^{4}\sum_{j=1}^{3} ij = \sum_{i=1}^{4}(i + 2i + 3i) = \sum_{i=1}^{4} 6i = 2 + 12 + 18 + 24 = 60$$

Sequences and Summations

Activity 1

***Write down the first five terms of the sequence whose general terms is given by a_n .**

1. **a)** $a_n = 5\text{-}2n$ **b)** $a_n = n^2 + n$

2. **a)** $a_n = 4 \cdot 3^n$ **b)** $a_n = \dfrac{n}{n+1}$

***Find the formula for the general term a_n of the sequence whose first five terms are:**

3. **a)** 1, 5, 9, 13, 17, … **b)** 72, 36, 18, 9, $\frac{9}{2}$, …

4. **a)** 0, 3, 8, 15, 24, … **b)** -1, -2, -3, -4, -5, …

5. Determine which of the following sequences is an arithmetic sequence; if it is an arithmetic sequence, what is *d*?

a) 3, 8, 13, 18, 23, … **b)** 2, 3, 5, 8, 12, …
c) 10, 9, 8, 7, 6, … **d)** 7, 1, -5, -11, -17, …

6. Determine which of the following sequences is a geometric sequences; if it is what is *r*?

a) 3, 2, $\frac{4}{3}$, $\frac{8}{9}$, $\frac{16}{27}$, … **b)** 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, …
c) 2, -2, 2, -2, 2, … **d)** 1, 4, 9, 16, 25, …

7. Define the following sequences, recursively

a) 3, 6, 12 , 24, 48, … **b)** 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, …
c) 2, -2, 2, -2, 2, … **d)** 1, 4, 9, 16, 25, …

8. What are the values of the following sums?

a) $\sum_{k=1}^{5}(k+1)$ **b)** $\sum_{j=0}^{4}(-2)^j$

9. Find the value of each of the following sums.

a) $\sum_{j=0}^{8}(1+(-1)^j)$ **b)** $\sum_{j=0}^{8}(3^j - 2^j)$

10. Make up arithmetic and geometric sequences, then find the sum of the first five (5) terms.

Sequences and Summations

Activity 2

1. A school auditorium has 12 seats in the front row, 13 seats in the second row, 14 seats in the third row, and so on. If there are 30 rows in the auditorium, how many seats are in the last row?

2. Suppose a sequence has first term 2 and third term 18.

a) If the sequence is arithmetic, find the common difference d, and list the first five term of the sequence.

b) If the sequence is geometrid, find two possible values for the common ratio r, and list the first five term of the geometric sequence in each case.

3. Describe the pattern of the Fibonacci sequence: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55,… and define it recursively.

4. Find the value of the following sum

$$\sum_{j=0}^{8}(1+(-1)^{j})$$

5. Compute the following double sum

$$\sum_{i=1}^{3}\sum_{i=1}^{2}(i-j)$$

Theme 1: Rational Numbers

Lesson 5: Binomials, Pascal Triangles, and the Binomial Theorem

Do You Know?

Binomial Expressions

Consider $(x+y)^3$. The expression in parentheses has two terms and so is known as ***binomial*** **expression.** Let us investigate what happens when we expand powers of a binomial expression. For the first example, recall that any quantity (other than 0 itself) raised to the power 0 is defined to be equal to 1.

$$(x+y)^0 = 1$$
$$(x+y)^1 = x + y = 1 \cdot x + 1 \cdot y$$
$$(x+y)^2 = x^2 + 2xy + y^2 = 1 \cdot x^2 + 2 \cdot xy + 1 \cdot y^2$$
$$(x+y)^3 = x^3 + 3x^2y + 3x\,y^2 + y^3 = 1 \cdot x^3 + 3 \cdot x^2y + 3 \cdot xy^2 + 1 \cdot y^3$$

What patterns do notice in the coefficients?

Figure A
Pascal's Triangle

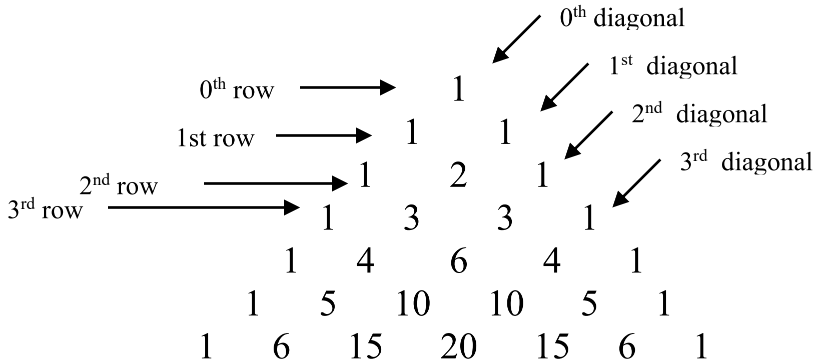
1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1

The array is named after the French mathematician Blaise Pascal (1623-1662) who showed that these numbers play an important role in mathematics. However, the triangle was certainly known in China as early as the twelfth century. Pascal's triangle is rich with remarkable patterns and is also extremely useful. Before discussing the patterns, we observe that it is customary to call the single 1 at the top of the triangle the 0^{th} row (since, for example, in the path counting problem discussed above, this 1 would represent a path length 0).

Binomials, Pascal Triangles, and the Binomial Theorem

For consistency, we will also call the initial 1in any row the 0^{th} element in the row, and the initial diagonal of 1s the 0^{th} diagonal. Thus, 1 is the zeroth element in the fourth row, 4 is the first element, 6 the second element, and so on.

Figure B: Numbered rows and diagonals in Pascal's Triangle



Observe that:

For $(x+y)^0$, the coefficient is	1
For $(x+y)^1$, the coefficients are	1 1
For $(x+y)^2$, the coefficients are	1 2 1
For $(x+y)^3$, the coefficients are	1 3 3 1

Numbers in Pascal's Triangle

Because of this connection with the coefficients in binomial expansions, the combination numbers C(*n, r*), that, as you remember, are the same numbers found in Pascal's Triangle, are also known as the **binomial coefficients**.

What in the world is the connection between combination numbers, which tell us the number of subsets of a certain size, and the coefficients we get when we expand a binomial expression? Perhaps we can gain some insight by looking more closely at exactly how we obtain those coefficients.

When we expand $(x+y)^2$ using the distributive property, we obtain

$$(x+y)^2 = (x+y)\,(x+y) = xx + x\,y + yx + yy = x^2 + 2\,x\,y + y^2 .$$

If we want a term with one x and one y, for example, we can choose the x from the $(x+y)$ factor on the left, which gives xy, or from the $(x+y)$ factor on the right, which gives yx. Once we choose where the x term comes from, we have only one choice for where the y term comes from. Thus the total number of ways to get a term with one x and one y is C(2, 1)·1=2·1=2, so 2 is the coefficient of the xy term in the simplified form.

Binomials, Pascal Triangles, and the Binomial Theorem

Let's try it with the third power.

$$\begin{aligned}(x + y)^3 &= (x + y)(x + y)(x + y)\\ &= (xx + xy + yx + yy)(x + y)\\ &= xxx + xxy + xyx + xyy + yxx + yxy + yyx + yyy\\ &= x^3 + 3x^2y + 3xy^2 + y^3\end{aligned}$$

To get an $x^2 y$ term, we must choose an x from two of the three factors of $(x + y)$ and a y from the remaining factor. If we choose x from the first two factors we get xxy, whereas choosing x from the first and third factors yields xyx, and choosing x from the last two factors gives yxx. As before, once we choose where the x's come from, we have only one choice for where the y comes from. Thus, there are C(3, 2)·1 = 3·1= 3 terms with two factors of x and one factor of y, and 3 is the coefficient of $x^2 y$.

From these examples, we can generalize this pattern for any non-negative integer power of a binomial expression, to obtain the following theorem. Before stating the theorem, we should point out that in each term of the binomial expansion, the sum of the powers of x and y is equal to the power of the binomial. For example, in the term $x^2 y$, the power of x is 2, the power of y is 1, and 2 + 1 = 3, which is the power of $(x + y)$ in this example. In general, if r is the power of x in a term of the expansion of $(x + y)^n$, then the power of y in that term must be $n - r$.

Theorem 1 The coefficient of $x^r y^{n-r}$ in the expansion of $(x + y)^n$ is $C(n, r)$ or more generally.

Binomial Theorem

Let x and y be variables, and let n be a positive integer. Then

$$\begin{aligned}(x + y)^n &= \sum_{j=0}^{n} C(n, j)\, x^{n-j} y^j\\ &= \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \cdots + \binom{n}{n-1} xy^{n-1} + \binom{n}{n} y^n .\end{aligned}$$

What is the expansion of $(x + y)^4$?

Solution: From the binomial theorem it follows that

$$\begin{aligned}(x + y)^4 &= \sum_{j=0}^{4} C(4, j)\, x^{4-j} y^j\\ &= C(4, 0)x^4 + C(4, 1)\, x^3 y + C(4, 2)\, x^2y^2 + C(4, 3)xy^3 + C(4,4)y^4\\ &= x^4 + 4x^3 y + 6x^2y^2 + 4xy^3 + y^4 .\end{aligned}$$

Binomials, Pascal Triangles, and the Binomial Theorem

Activity 1

***In the expansion of $(x + y)^{10}$ what is the coefficient of**

1. **a)** x^9y **b)** xy^9

2. **a)** x^6y^4 **b)** x^4y^6

3. **a)** x^3y^7 **b)** x^7y^3

4. **a)** x^5y^5 **b)** x^2y^8

5. Expand $(a - b)^4$ completely

6. Expand $(x - 2)^5$ completely

7. Expand $(35 - 2t)^3$ completely

8. Find the coefficient of x^4y^4 in the binomial expansion of $(x + 3y)^9$.

9. In what rows of the Pascal Triangle does one find a given number appearing only once?

10. What patterns do you see regarding the numbers in the diagonals from the outer and inner right and the corresponding diagonals from the outer and inner left.

Binomials, Pascal Triangles, and the Binomial Theorem

Activity 2

1. Find a pattern in the Row of Sums of Pascal's Triangle

(a) Compute the sum of the elements in each of rows zero through four of Pascal's triangle.

(b) Look for a pattern in the results of part (a) and guess a general rule.

(c) Use Figure A to check your guess for rows five through eight.

(d) Give a convincing argument that your guess in part (b) is correct.

2. Let *S* be a set with 8 elements. Use Pascal's Triangle to answer the following Questions.

a) How many subsets of *S* have exactly 2 elements?

b) How many subsets of *S* have exactly 3 elements?

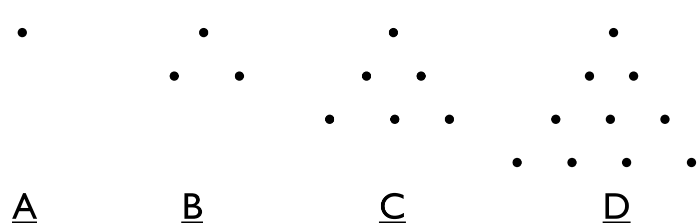
c) How many subsets of *S* have exactly 4 elements?

d) How many subsets of *S* have exactly 5 elements?

e) How many subsets of *S* are there altogether?

3. There are ten (10) books on Abdou's reading list. Over the summer, he must read four (4) of them. In how many ways can he choose four (4) books to read?

4. Study the Pascal Triangle. Do you see any pattern in some of the triangle numbers (the numbers that form sub-triangles):



Theme: II
Real Numbers: Decimals and Percents

Theme II: Real Numbers: Decimals and Percents

Lesson 1: Real Numbers and Inequalities

Do You Know?

Real Numbers

When expressed in decimal form, rational numbers are either terminating decimals with a finite number of digits (such as 0.25, which is $\frac{1}{4}$) or repeating decimals in which a pattern repeats over rand over (such as 0.333…, which is $\frac{1}{3}$).

Irrational numbers are numbers that **cannot** be expressed in the form x/y. when written as decimals, irrational numbers neither terminate nor have repeating pattern. For example, the number $\sqrt{2}$ is irrational because it cannot be expressed exactly in a form x/y ; as a decimal, we can write it as 1.414213565…, where the dots mean that the digits continue forever with no pattern. The number π is also an irrational number, which as a decimal is written 3.14159265….

The set of **real numbers** consists of both rational and irrational numbers ; hence it is represented by the entire number line. Each point on the number line has a corresponding real number, and each real number has a corresponding point on the number line. In other words, the real numbers are the integers and "everything in between." A few selected real numbers are shown on the number line which is presented.

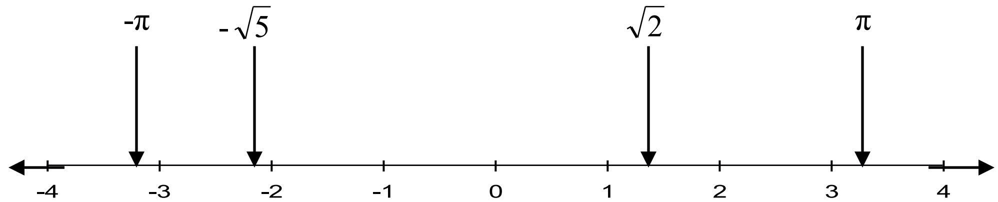
Examples :

- The number 25 is a natural number, which means it is also a whole number, an integer, a rational number, and a real number.
- The number -6 is an integer, which means it is also a rational number and a real number.
- The number $\frac{2}{3}$ is a rational number, which means it is also a real number.
- The number 7.98418…is an irrational number ; the dots indicate that the digits continue forever with no particular pattern. It is also a real number.

Real Numbers and Inequalities

The Real Line and Inequalities

The set of real numbers, denoted by **R**, plays a dominant role in mathematics. We assume the reader is familiar with the geometric representation of **R** by means of the points on a straight line. As shown below, a point, called the **origin**, is chosen to represent) and another point, usually to the right of 0, to represent 1. Then there is a natural way to pair off the points on the line and the real numbers, i.e. each point will represent a unique real number and each real number will be represented by a unique point. For this reason we refer to **R** as the **real line** and use the words point and number interchangeably.



POSITIVE NUMBERS

Those numbers to the right of 0 on the real line **R**, i.e. on the same side as 1, are the **positive numbers**; those numbers to the left of 0 are the **negative numbers**. The set of positive numbers can be completely described by the following axioms:

[P_1] If $a \in$ **R**, then exactly one of the following is true: a is positive; $a = 0$; $-a$ is positive.
[P_2] If $a, b \in$ **R** are positive, then their sum $a + b$ and their product $a \cdot b$ are also positive.
It follows that a is positive if and only if $-a$ is negative.

Example: We show, using only **[P_1]** and **[P_2]**, that the real number 1 is positive.
By **[P_1]**, either 1 or -1 is positive. If -1 is positive then, by **[P_2]**, the product (-1) (-1) = 1 is positive. But this contradicts **[P_1]** which states that 1 and -1 cannot both be positive. Hence the assumption that -1 is false and so 1 is positive.

Example: The real number -2 is negative. For, by the preceding example, 1 is positive and so, by **[P_2]**, the sum 1 + 1 = 2 is positive; hence -2 is not positive, i.e. -2 is negative.

Real Numbers and Inequalities

ORDER

An **order relation** in **R** is defined using the concept of positiveness.

Definition: The real number a is **less than** the real number b, written $a < b$, if the difference $b - a$ is positive.

The following notation is also used:

$a > b$, read a is greater than b, means $b < a$
$a \le b$, read a is less than or equal to b, means $a < b$ or $a = b$
$a \ge b$, read a is greater than or equal to b, means $b \le a$

Geometrically speaking,

$a < b$ means a is to the left of b on the real line **R**
$a > b$ means a is to the right of b on the real line **R**

Examples: $2 < 5$, $-6 \le -3$, $4 \le 4$, $5 > -8$

Example: A real number x is positive if $x > 0$, and x is negative if $x < 0$. For $x \ne 0$, x^2 is always greater than 0: $x^2 > 0$.

Example: The notation $2 < x < 5$ means $2 < x$ and also $x < 5$; hence x will lie between 2 and 5 on the real line.

We refer to the relations $<$, $>$, $\le$ and $\ge$ as **inequalities** in order to distinguish them from the equality relation $=$. We also shall refer to $<$ and $>$ as strict inequalities. We now state basic properties about inequalities which shall be used throughout.

Theorem A: Let a, b and c be real numbers.

(i) The sense of an inequality is not changed if the same real number is added to both sides:
If $a < b$, then $a + c < b + c$.
If $a \le b$, then $a + c \le b + c$.

(ii) The sense of an inequality is not changed if both sides are multiplied by the same positive real number:
If $a < b$ *and* $c > 0$, then $ac < bc$.
If $a \le b$ and $c > 0$, then $ac \le bc$.

(iii) The sense of an inequality is reversed if both sides are multiplied by the same *negative* number:
If $a < b$ *and* $c < 0$, then $ac > bc$.
If $a \le b$ and $c < 0$, then $ac \ge bc$.

Real Numbers and Inequalities

(FINITE) INTERVALS

Let a and b be real numbers such that $a < b$. Then the set of all real numbers x satisfying

> $A < x < b$ is called the **open interval** from a to b
> $A \leq x \leq b$ is called the **closed interval** from a to b
> $A < x \leq b$ is called the **open-closed** interval from a to b
> $A \leq x < b$ is called the **closed-open** interval from a to b

The points a and b are called the **end points** of the interval. Observe that a closed interval contains both its endpoints, an open interval contains neither endpoint, and an open-closed and a closed-open interval contains exactly one of its endpoints:

Open interval: $a < x < b$

closed interval: $a \leq x \leq b$

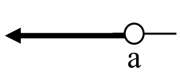
Open-closed interval: $a < x \leq b$

Closed-open interval: $a \leq x < b$

The open-closed and closed-open intervals are also referred to as being half-open (or: half-closed).

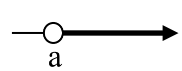
INFINITE INTERVALS

Le a be any real number. Then the set of all real numbers x satisfying $x < a$, $x \leq a$, $x > a$ or $x \geq a$ is called and infinite interval.

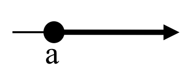


$x < a$

$x \leq a$



$x > a$



$x \geq a$

Real Numbers and Inequalities

LINEAR INEQUALITIES IN ONE UNKNOWN

Every linear inequality in one unknown x can be reduced to the form

$ax < b,\ ax \le b,\ ax > b$ or $ax \ge b$

If $a \ne 0$, then both sides of the inequality can be multiplied by 1/a with the sense of the inequality reversed if a is negative. Thus the inequality can be further reduced to the form

$x < c.\ x \le c, x > c$ or $x \ge c$

Whose solution set is an infinite interval.

Example: Consider the inequality $5x + 7 \le 2x + 1$. Adding $-2x - 7$ to both sides (or: transposing), we obtain
$5x - 2x \le 1 - 7$ or $3x \le -6$
Multiplying both sides by $\frac{1}{3}$ (or dividing both sides by 3) we finally obtain

$x \le -2$

Example: Consider the inequality $2x + 3 < 4x + 9$. Transposing, we obtain
$2x - 4x < 9 - 3$ or $-2x < 6$

Multiplying both sides of the inequality by $-\frac{1}{2}$ (or: dividing both sides by -2) and reversing the inequality since $-\frac{1}{2}$ is negative, we finally obtain

$x > -3$

ABSOLUTE VALUE

The **absolute value** of a real number x, written $|x|$, is defined by

$$|x| = \begin{cases} x & if \quad x \ge 0 \\ -x & if \quad x < 0 \end{cases}$$

That is, if x is non-negative then $|x| = x$, and if x is negative then $|x| = -x$. Thus the absolute value of every real number is non-negative: $|x| \ge 0$ for every $x \in \mathbf{R}$.

Real Numbers and Inequalities

Geometrically speaking, the absolute value of x is the distance between the pont x on the real line and the origin, i.e. the point 0. Furthermore, the distance between any two points a, b $\in$ **R** is $|a-b| = |b-a|$.

Example: $|-2| = 2$, $|7| = 7$, $|-\pi| = \pi$, $|-\sqrt{2}| = \sqrt{2}$

Example: $|3-8| = |-5| = 5$ and $|8-3| = |5| = 5$

Example: The statement $|x|$, 5 can be interpreted to mean that the distance between x and the origin is less than 5; hence x must lie between -5 and 5 on the real line. In other words,

$$|x| < 5 \text{ and } -5 < x < 5$$

Have identical meaning and, similarly,

$$|x| \le 5 \text{ and } -5 \le x \le 5$$

Have identical meaning.

The central facts about the absolute value are the following:

Theorem B: Let a and b be any real numbers. Then:

(i) $|a| \ge 0$, and $|a| = 0$ if $a = 0$

(ii) $-|a| \le a \le |a|$

(iii) $|ab| = |a| \cdot |b|$

(iv) $|a+b| \le |a| + |b|$

(v) $|a+b| \ge |a| - |b|$

Real Numbers and Inequalities

Activity 1

1. List six (6) different real numbers; three (3) rationals and three (3) irrationals.

Write each statement using inequality notations :

2. a. a is less than b. b. a is greater than b.

3. c. a is not greater than b. d. a is not less than b

4. e. a is less than or equal to b f. a is not greater than or equal to b

Rewrite the following geometic relationships between the given real numbers using the inequality notation:

5. a. y lies to the right of 8 b. x lies between -3 and 7

6. c. z lies to the left of -3 d. w lies between 5 and 1

Describe and diagram each of the following intervals:

7. a. $2 < x < 4$ b. $-1 \leq x \leq 2$ c. $x > -1$

8. d. $-3 < x \leq 1$ e. $-4 \leq x < -1$ f. $x \leq$

Absolute Value

Evaluate:

9. a. $|3-5|$ b. $|-3+5|$ c. $|3-7|$ - $|-5|$

Rewrite without the absolute value sign:

10. a. $|x| \leq 3$ b. $|x-2| < 5$ c. $|2x-3| \leq 7$

Real Numbers and Inequalities

Activity 2

Solve each inequality

1. a. $3 < 2x - 5 < 7$ b. $-7 \leq -2x + 3 \leq 5$

Absolute Value

Evaluate:

2. a. $|3-5|$ b. $|-3+5|$ c. $|-3-5|$ d. $|3-7|$ - $|-5|$

3. Solve: $\frac{1}{2}x + \frac{x-2}{3} < 2x - \frac{1}{12}$

Solve each inequality and diagram its solution set:

4. a. $3x - 1 \geq 4x + 2$
   b. $x - 3 > 1 + 3x$
   c. $2x - 3 \leq 5x - 9$

Rewrite without the absolute value sign:

5. a. $|x| \leq 3$ b. $|x-2| < 5$ c. $|2x-3| \leq 7$

Theme II: Real Numbers: Decimals and Percents

Lesson 2: Powers of Ten (10) and Computation with Decimals

Do You Know?

Powers of 10 indicate how many times to multiply 10 by itself. For example:

$10^2 = 10 \times 10 = 100$

$10^6 = 10 \times 10 \times 10 \times 10 \times 10 \times 10 = 1{,}000{,}000$

Negative powers indicate reciprocals of corresponding positive powers. For example:

$$10^{-2} = \frac{1}{10^2} = \frac{1}{100} = 0.01$$

$$10^{-6} = \frac{1}{10^6} = \frac{1}{1{,}000{,}000} = 0.000001$$

Thus, powers of 10 follow two basic rules:

1. A positive exponent tells how many 0s follow the 1. For example, 10^0 is a 1 followed by no 0s; 10^8 is a 1 followed by eight 0s.
2. A negative exponent tells how many places are to the right of the decimal point, including the 1. For example, $10^{-1} = 0.1$ has one place to the right of the decimal point; $10^{-6} = 0.000001$ has six places to the right of the decimal point.

Multiplying and Dividing Powers of 10 Multiplying powers of 10 simply requires adding exponents $10^n \times 10^m = 10^{n+m}$

Examples: A. $10^4 \times 10^7 = 10{,}000 \times 10{,}000{,}000$

$10^{11} = 100{,}000{,}000{,}000$

B. $10^5 \times 10^{-3} = 100{,}000 \times 0.001$

$10^2 = 100$

C. $10^{-8} \times 10^{-5} = 0.00000001 \times 0.00001$

$10^{-13} = 0.0000000000001$

Powers of Ten (10) and Computation with Decimals

Dividing powers of 10 requires subtracting exponents. $\frac{10^n}{10^m} = 10^{n-m}$ For example:

Examples: C. $\frac{10^5}{10^3}$ = 100,000 $\div$ 1000 = 100 = 10^2

D. $\frac{10^3}{10^7}$ = 1000 $\div$ 10,000,000 = 0.0001 = 10^{-4}

E. $\frac{10^{-4}}{10^{-6}}$ = 0.0001 $\div$ 0.000001 = 10^2

Powers of Powers of 10
We can use the multiplication and division rules to raise powers of 10 to other powers. For Example:

F. $(10^4)^3 = 10^4 \times 10^4 \times 10^4 = 10^{4+4+4} = 10^{12}$

Adding and Subtracting Powers of 10 There is no shortcut for adding or subtracting powers of 10, as there is for multiplication and division. The values must be written in longhand notation.

Examples: G. $10^6 + 10^2$ = 1,000,000 +100
=1,000,100

H. $10^8 + 10^{-3}$ = 100,000,000 + 0.001
= 100,000,000.001

I. $10^7 - 10^3$ = 10,000,000 – 1000
= 9,999,000

$(10^n)^m = 10^{n \times m}$

Decimals Since our number system is a positional system based on ten {0,1,2,3,4,5,6,7,8,9} numerals, we have used the term decimal system. In general, we refer to expressions like 0.235 or 2.7142 as decimals as opposed ot 24, 98, 0 or 2478, which we more often speak of as whole numbers or integers. In fact, both are part and parcel of the same system. Just as the expanded from of 2478 is

$$2478 = 2 \cdot 10^3 + 4 \cdot 10^2 + 7 \cdot 10^1 + 8 \cdot 10^0$$
$$= 2000 + 400 + 70 + 8$$

The expanded form of 0.235 is

$$0.235 = 2 \cdot \frac{1}{10^1} + 3 \cdot \frac{1}{10^2} + 5 \cdot \frac{1}{10^3}$$

And the expanded form of 23.47 is

$$23.47 = 2 \cdot 10^1 + 3 \cdot 10^0 + 4 \cdot \frac{1}{10^1} + 7 \cdot \frac{1}{10^2}$$
$$= 20 + 3 + \frac{4}{10} + \frac{7}{100}$$

Powers of Ten (10) and Computation with Decimals

Adding and Subtracting Decimals

Suppose we wish to add 2.71 and 32.762, we would do the following:

2.71 + 37.762 = 40.472

To add decimals by hand, write the numbers in the vertical style lining up the decimal points of each number over the decimal point of the other number and then add essentially just as we add integers.

Suppose now that we want to subtract 2.71 from 37.762, we would do the following:

37.762 – 2.71 = 35.052

Thus, we write the problem in vertical style lining up the decimal points and then subtract essentially as we subtract the integers.

Multiplying Decimals

Suppose we wish to calculate the product (31.76) · (4.6), we would do the following:

$$(31.76) \cdot (4.6) = 146.096$$

Or

$$\begin{array}{r} 31.76 \\ \times \quad 4.6 \\ \hline 19\,056 \\ 127\,04 \\ \hline 146.096 \end{array}$$

<u>To multiply two decimals:</u>
1. Multiply as with integers.
2. Count the number of digits to the right of the decimal point in each number in the product, add these numbers, and call their sum *t*.
3. Finally, place the decimal point in the product obtained so that there are *t* digits to the right of the decimal point.

Powers of Ten (10) and Computation with Decimals

Dividing Decimals

Suppose we want to divide 537.6 by 2.56, we have

$$537.7 \div 2.56 = 210$$

The problem is reduced to that of dividing 52,760 by 256; that is, to dividing integers. Recall that, when confronted by a division like

$$2.56\overline{)537.6},$$

students are often told to « move the decimal point in both the divisor and the dividend 2 places to the right so that the divisor becomes an integer." Multiplying both by justifies this rule and, by hand, we have

```
          210.
2.56)537.60.
       512
       256
       256
         0
```

We check our answer by multiplying the divisor times the quotient.

```
Check:  2.56
          210
       ------
         2560
       × 512
       ------
       537.60
```

Nonterminating Decimals and Rational Numbers

Somewhat surprisingly not all national number have decimal expansions terminated. For example, it is well-known that

$$\frac{1}{3} = 0.333\ \ldots - 0.\overline{3}$$

where the three dots indicate that the decimal continues *ad infinitum* and the bar over the 3 indicates the digit or group of digits that repeats.

A nonterminating decimal that has the property that a digit or group of digits repeats *ad infinitum* from some point on is called a **periodic** or **repeating decimal**. The number of digits in the repeating group is called the **length of the period**.

Every repeating decimal represents a rational number a/b. If a/b is in simplest form, b must contain a prime factor other than 2 or 5. Conversely, if a/b is such a rational number its decimal representation must be repeating.

Powers of Ten (10) and Computation with Decimals

Ordering Decimals

Ordering decimals is much like ordering integers. For example, to determine the larger of 247,761 and 2,326,447 write both numerals as if they had the same number of digits; that is, write

0,247,761 and 2,326,447

Then determine the first place from the *left* where the digits differ. It follows from the idea of positional notation that the larger integer is the integer with the larger of these two different digits. In the present case, the first digits differ and so

$$0,247,761 < 2,326,447$$

In a similar example,

$$34,716 < 34,723$$

Since the first pair of corresponding digits that differ are the 1 and the 2 and $1<2$.

To order two positive decimals, determine the first digitis from the left that differ. The decimal with the lesser of these two digits is the lesser decimal.

Example Decide which of the decimals represent the lesser number.

23.45 and 23.4545

Solution Since 23.45 = 23.4545….., the first digits from the left differ are 0 and 5. Since $0 < 5$, it follows that $23.45 < 23.4545$

Powers of Ten (10) and Computation with Decimals

Activity 1

*Given a) 10^3 b) 10^4 c) 10^5 d) 10^9 e) 10^{12}
f) 10^{-3} g) 10^{-4} h) 10^{-5} i) 10^{-9} j) 10^{-12}

1. In (*) express as a single integer or a single decimal.

   a), d), g), and h)

2. In (*) add the powers

   b) + d) and c) + h)

3. In (*) subtract the powers

   c) – b) and a) – g)

4. In (*) multiply

   e) · i) and f) · h)

5. In (*) divide

   j) ÷ c) and h) ÷ j)

6. In (*) raise c) to the 4^{th} power.

7. Make a mental calculation to determine an approximate answer and then determine the accurate result of each of these calculations.

   a) 23.47 + 7.81 b) 351.42 – 417.815

8. Calculate these products.

   a) (471.2) · (2.3) b) (36.34) · (1.02)

9. Divide 36.9 by 1.23 and check your answer.

10. Write each of these repeating decimals in the form *a*/*b* where *a* and *b* are integers and the fraction is in the simplest form. Check by dividing *a* and *b* with your calculator.

Powers of Ten (10) and Computation with Decimals

Activity 2

1. Arrange these numbers in order from least to greatest

   $\frac{11}{24}$, $\frac{3}{8}$, 0.37, 0.4584, 037666…, 0.4583

2. Express these two real numbers in expanded form

   a) 54,312   b) 21.345

3. Fill in the blanks so that each of these is an arithmetic progression.
   a) 3.4, 4.3, 5.2, _____, _____, ______
   b) -31.56 _____, -21.10, _____, ______,______,______
   c) 0.0114, _____,______,0.3204,______,______
   d) 1.07, _____,______,______,8.78,_____

4. Fill in the blanks so that each of these is a geometric progression.
   a) 2.11,2.327,_____,_____,_____
   b) 35.1,_____,2.835,_____,_____
   c) 6.01,_____,_____,0.75125,______

5. Use long division to find the decimal expansion of $\frac{3}{7}$. What are the repeating decimals?

Theme II: Real Numbers: Decimals and Percents

Lesson 3: Ratio and Proportion

Do You Know?

Ratio

At basketball practice, Caralee missed 18 free throws out of 45 attempts. Since she mad 27 free throws, we say that the **ratio** of the number missed to the number made was 18 to 27. This can be expressed by the fraction 18/27 or, somewhat archaically, by the notation 18 : 27. We will always use the fraction notation in what follows.

Other ratios from Caralee's basketball practice are:

- The ratio of the number of shots made to the number attempted – 27 / 45,
- The ratio of the number of shots missed to the number attempted – 18 / 45,
- The ratio of the number of shots made to the number missed – 27 / 18.

If *a* and *b* are real numbers with $b \neq 0$, the **ratio of *a* to *b*** is the fraction *a / b*.

Ratios occur with great frequency in everyday life. If you use 30.4 liters of gasoline in driving 400.4 miles, the efficiency of your car is measured in miles per gallon given by the ration 400.4 / 10.4 or 38.5 miles per gallon. If Lincoln Grade School has 405 students and 15 teachers, the student teacher ratio is the quotient 405/15. If Jose Varga got 56 hits in 181 times at bat, his batting average is the ration 56/181. The number of examples that could be citied is almost endless.

Ratio and Proportion

Determining Ratios

Determine these ratios.

a. The ratio of the number of boys to the number of girls in Martin Luther King High School if there are 285 boys and 228 girls.
b. The ratio of the number of boys to the number of students in part (a)

Solution

a. The desired ratio is 285 / 228.
b. Since the total number of students is 285 + 228 (that is, 513), the desired ratio is 285 / 513.

The ratio of the number of boys to the number of students in Martin Luther King High School was shown to be 285 / 513 or 285 to 513. This is certainly correct, but it is not nearly as informative as it would be if the ratio were written in simplest form.

Thus,

$$\frac{285}{513} = \frac{5}{9}$$

$285 \div 57 = 5$
$513 \div 57 = 9$

and this says that 5 / 9 (or a little more than 1 / 2) of the students in Martin Luther King High School are boys. Reducing a ratio to lowest or simplest terms is often useful and informative.

Expressing Ratios in Simplest Form

Express this ratio in simplest form.

The ratio of 385 to 440

Solution: The ratio of 385 to 440 is the quotient 385 / 440. Expressing this in simplest form, we have

$$\frac{385}{440} = \frac{7}{8}$$

Ratio and Proportion

Determining a Less Obvious Ratio

If one seventh of the students at John F. Kennedy High School are non-swimmers, what is the ratio of non-swimmers to swimmers?

Solution

The desired ratio is the number of non-swimmers divided by the number of swimmers. Can we determine these numbers from the information given? Actually, no; but the problem can still be solved. Supposed there are n students in the school. Then $\frac{n}{7}$ are non-swimmers and $\frac{6n}{7}$ are swimmers. Thus, the desired ratio is

$$\frac{\frac{n}{7}}{\frac{6n}{7}} = \frac{n}{7} \cdot \frac{7}{6n} = \frac{1}{6}.$$

Proportion

Ratios allow up to make clear comparisons when actual numbers sometime make them more obscure. For example, at basketball practice, Caralee made 27 of 45 free throws attempted and Sonja made 24 of 40 attempts. Which player appears to be the better foul shot shooter? For Caralee, saying that the ratio of shots made to shots tried is 27 / 45 amounts to saying that she made 3 /5 of her shots. That is,

$$\frac{27}{45} = \frac{3}{5}.$$

Similarly, for Sonja the ratio of shots made to shots attempted is

$$\frac{24}{40} = \frac{3}{5},$$

And this suggests that the two girls are equally capable at shooting foul shots. Because of its importance in such comparisons, the equality of two ratios is called a **proportion**.

Definition: If a/b and c/d are two ratios and

$$\frac{a}{b} = \frac{c}{d},$$

This equality is called a **proportion.**

Ratio and Proportion

We know that

$$\frac{a}{b} = \frac{c}{d}$$

For integers *a, b, c,* and *d*, if, and only if, $ad = bc$. But essentially the same argument holds if *a, b, c*, and *d* are real numbers. This leads to the next theorem.

Theorem: Conditions for a Proportion
The equality

$$\frac{a}{b} = \frac{c}{d}$$

is a proportion if, and only if, $ad = bc$.

Determining Proportions
Determine x so that the equality is proportion.

$$\frac{28}{49} = \frac{x}{21}$$

Solution We use the preceding theorem which amounts to multiplying both sides of the equality by the product of the denominators or "cross multiplying" as we often say.

$$\frac{28}{49} = \frac{x}{21}$$

$$28 \cdot 21 = 49x$$

$$\frac{28 \cdot 21}{49} = x$$

$$12 = x$$

Thus $\frac{28}{49} = \frac{12}{21}$

Therefore, $\frac{a}{b} + 1 = \frac{c}{d} + 1,$ {add 1 to both sides}

$\frac{a}{b} + \frac{b}{b} = \frac{c}{d} + \frac{d}{d}$ {since $\frac{b}{b} = 1 = \frac{d}{d}$ }

and $\frac{a+b}{b} = \frac{c+d}{d}$ {add fractions}

Ratio and Proportion

Applications of Proportions

Suppose that a car is traveling at a constant rate of 55 miles per hour. Table 7.2 gives the distances the car will travel in different time periods.

Table A

Distance Traveled in *t* Hours at 55 Miles per Hour				
t = time	*d* = distance		*t* = time	*d* = distance
1	55		5	275
2	110		6	330
3	165		7	385
4	220		8	440

The ratios *d/t* are all equal for the various time periods shown. That is,

$$\frac{55}{1}=\frac{110}{2}=\frac{165}{3}=\frac{220}{4}=\frac{275}{5}=\frac{330}{6}$$

and so on. Thus, each pair of ratios from the list form a proportion. Indeed, $d/t = 55$ for every pair d and t. This is also expressed by saying that the distance traveled at a constant rate is proportional to the elapsed time. In the above instance

$$d = 55t$$

for every pair d and t. The number 55 is called the **constant of proportionality.**

Definition *y* Proportional to *x*

If the variables x and y are related by the equation

$$y = kx\text{, which is the same } \frac{y}{x} = k$$

then **y is said to be proportional to x** and k is called the **constant of proportionality.**

This situation is extremely common in everyday life.
Gasoline consumed by your car is proportional to the miles traveled.
The cost of pencils purchased is proportional to the number of pencils purchased.
Income from the school raffle is proportional to the number of tickets sold, and so on.

Ratio and Proportion

Activity 1

*** There are 30 girls and 24 boys in an 8^{th} grade class.**

1. What is the ratio of
   a) boys to girls b) girls to students

2. What is the ratio of
   a) boys to students b) girls to boys

3. What is the ratio of
   a) students to girls b) students to boys

4. Determine which of the following are proportions
   a) $\frac{2}{3} = \frac{8}{12}$ b) $\frac{21}{28} = \frac{27}{36}$ c) $\frac{7}{28} = \frac{8}{31}$

5. Determine which of the following are proportions:
   a) $\frac{51}{85} = \frac{57}{95}$ b) $\frac{14}{49} = \frac{18}{60}$ c) $\frac{20}{55} = \frac{28}{48}$

6. Express each of these ratios in the simplest form.
   a) a ratio of 24 to 16 b) a ratio of 296 to 111
   c) a ratio of 248 to 372 d) a ratio of 209 to 341

7. Determine x so that the equality is a proportion
   a) $\frac{35}{63} = \frac{20}{x}$ b) $\frac{2.11}{3.49} = \frac{1.7}{x}$

8. Determine the value of r so that each is a proportion
   a) $\frac{6}{14} = \frac{r}{21}$ b) $\frac{8}{12} = \frac{10}{r}$

9. Determine values of s and t so that each is a proportion
   a) $\frac{51}{t} = \frac{85}{95}$ b) $\frac{47}{3.2} = \frac{s}{7.8}$

10. When y is proportional to x^3 and y=32 when x=12; determine the value of y when x=6.

Ratio and Proportion

Activity 2

1. If s is proportional to t and $s = 62.5$ when t=7, what is s when t=10?

2. The flag pole at Sunnyside Elementary School casts a shadow 9'8'' long at the same time Mr. Schaal's shadow is 3'2''. If Mr. Schaal is 6'3'' tall, how tall is the flag pole to the nearest foot?

3. A kilometer is a bit more than six tenths of a mile. If the speed limit along a stretch of highway in Canada is 90 kilometers per hour, about how fast can you travel in miles per hour and still not break the speed limit?

4. a) If y is proportional to x^2 and $y = 27$ when $x = 6$, determine y when x =12.
   b) Determine the ratio of the y-values in part (a).
   c) If y and x are related as in part (a), what happens to the value of y if the value of x is doubled? Explain.

5. If y is proportional to $1/x$ and y=3.5 when x =84, determine y when x=14.
   (Hint: y=k(1/x).)

Theme II: Real Numbers: Decimals and Percents

Lesson 4: Percent

Do You Know?

One of the most important uses of ratios in school mathematics is the notion of percent per hundred. Thus, 50% is the ratio 50/100, and this is quickly reduced to the fraction ½ or written as the decimal 0.50. Thus, if I have 98$ and give you 50% of what I have, I give you

$$\frac{1}{2} \cdot \$98 = \$49 \qquad \text{or} \qquad 0.5 \times \$98 = \$49.$$

The ‘‘of ’’ in the preceding sentence translates into “times.” Thus,

50% of	means	50%×,
$\frac{1}{2}$ of	means	1/2×,
0.5 of	means	0.5×.

If r is any nonnegative real number, then r percent, written r %, is the ratio

$$\frac{r}{100}.$$

Since *r* %is defined as the ratio *r*/100 and dividing by 100 moves the decimal point 2 places to the left, it is easy to write a given percent as a decimal. For example,
12% = 0.12, 25% = 0.25, 130% = 1.3, and so on. Conversely, to write a decimal as a percent, we need only move the decimal point 2 places to the right.

Thus, 0.125 = 12.5% or $12\frac{1}{2}$% 0.10 = 10%, 1.50% and so on.

Expressing Decimals as Percents

Example Express these decimals as percents.
(a) 0.25 (b) 0.333 … (c) 2.15

SOLUTION

(a) 0.25 = 25%
(b) 0.33 … = 33.333 …%
(c) 1.255 = 125.5%

Percent

Expressing Percents as Decimals

Example Express these percents as decimals.

(a) 40% (b) 12% (c) 127%

Solution (a) 40% = 0.40
(b) 12% = 0.40
(c) 127%=1.27

Expressing Percents as Fractions

Example Express each of these percents as fraction in lowest terms.

(a) 60% (b) 66 2/3% (c) 12.5%

Solution (a) By definition 60% means 60/100. Therefore,

$$60\% = \frac{60}{100} = \frac{3}{5}.$$

(b) Here

$$37\frac{1}{2}\% = \frac{37\frac{1}{2}}{100}$$
$$= \frac{\frac{65}{2}}{100}$$
$$= \frac{65}{200} = \frac{13}{40}.$$

(c) $66\frac{2}{3}\% = \frac{66\frac{2}{3}}{100} = \frac{\frac{200}{3}}{100} = \frac{2}{3}.$

(d) $125\% = \frac{125}{100} = \frac{5}{4} = \text{or } 1\frac{1}{4}.$

Percent

Expressing Fractions as Percents

Example Express these fractions as percents.

(a) $\frac{1}{8}$ (b) $\frac{1}{3}$

Solution I

(Using proportions) Since percents are rations, we can use variables to determine the desired percents.

(a) Suppose 1/8 = *r*% = *r*/100. Then

$$r = 100 \cdot \frac{1}{8} = 12.5$$

and

$$r\% = 12.5\%$$

(b) Let 1/3 = *s* % = *s*/100. Then

$$s = \frac{100}{3} = 33\frac{1}{3} = 33.\bar{3}$$

3 3

and $$s\% = 33.\bar{3}\,\%$$

Solution II

(Using Decimals) Here we write the fractions as decimals and then percents.

(a) By division

$$\frac{1}{8} = 0.125 = 12.5\%$$

(b) Here

$$\frac{1}{3} = 0.333\ \ldots = 33.\bar{3}\,\% = 33\frac{1}{3}\,\%.$$

Percent

Applications of Percent

Use of percents is commonplace. Three of the most common types of usages are illustrated in the next three examples.

Calculating a Percentage of a Number

The Smetanas bought a house for \$175,000. If a 15% down payment was required, how much was the down payment?

Solution I

(Using an equation) The down payment is 15% of the cost of the house. Thus, if *d* is the down payment,

$$\begin{aligned} d &= 15\% \times \$175{,}000 \\ &= 0.15 \times \$175{,}000 \\ &= \$26{,}250 \end{aligned}$$

Solution II

(Using ratio and proportion) The ration of 23 to the number of questions on the test must be the same ratio as 92%. So,

$$\frac{23}{n} = \frac{92}{100} = 0.92.$$

Thus,

$$23 = 0.92 \times n$$

and

$$n = 23 \div 0.92 = 25$$

as before.

Calculating What Percentage One Number is of Another

Tara got 28 out of 35 possible points on her last math test. What percentage score did the teacher record in her grade book for Tara?

Solution I

(Using the definition) Tara got twenty-eight thirty fifths of the test right. Since

$$\frac{28}{35} = 0.80 = 80\%$$

the recorded 80% in her grade book.

Solution II

Let x be the desired percentage, then

$$\frac{x}{100} = \frac{28}{35}$$

and

$$x = \frac{28 \cdot 100}{35} = 80\%.$$

Compound Interest

If you keep money in a savings account at a bank, the bank pays you interest at a fixed rate (percentage) for the privilege of using your money. For example, suppose you invest \$5000 for a year at a 7% interest. How much is your investment worth at the end of the year? Since the interest earned is 7% of \$5000, the interest earned is

$$\begin{aligned} 7\% \times \$5000 &= 0.07 \times \$5000 \\ &= \$5000 \cdot (1.07) \\ &= \$5350. \end{aligned}$$

If you leave the total investment in the bank, its value at the end of second year is

$$\begin{aligned} \$5350 + 0.07 \times \$5350 &= \$5350 \cdot (1.07) \\ &= \$5000 \cdot (1.07)(1.07) \\ &= \$5000 \cdot (1.07)^2 \\ &= \$5724.50 \end{aligned}$$

Similarly, at the end of the third year, your investment would be worth

$$\$5000 \cdot (1.07)^3 = \$6125.22$$

to the nearest penny. In general, it would be worth

$$\$5000 \cdot (1.07)^n$$

at the end of n years. This is an example of **compound interest** where the term "compound" implies that each year you earn interest on all the interest earned in preceding years as well as on the original amount invested (the principal).

Usually, these days, interest is compounded more than once a year. Suppose the \$5000 investment just discussed was made in a bank that paid interest at the rate of 7% compounded semi-annually; that is twice a year. Since the rate for a year is 7%, the rate for half a year is 3.5%. Thus, the value of the investment at the end of the year (that is, at the end of *two* interest periods) is

$$\$5000(1.035)^2 = \$5356.13$$

and the values at the end of 2 years and 3 years respectively are

$$\$5000(1.035)^4 = \$5737.62$$

and

$$\$5000(1.035)^6 = \$6146.28.$$

Percent

Compounding more and more frequently is to your advantage and, to attract customers, some banks are now compounding monthly are even daily. The above calculations are typical, and are summarized in this theorem.

Theorem

Compounding Compound Interest
The value of an investment of *P* dollars at the end of *n* years if interest is paid at the annual rate of *r*% compounded *t* times a year is

$$P\left(1+\frac{r/t}{100}\right)^{nt} .$$

Percent

Activity 1

***Write each of the following rations as percents.**

1. a) $\frac{3}{16}$ b) $\frac{7}{25}$ c) $\frac{37}{40}$

2. a) $\frac{5}{6}$ b) $\frac{3.24}{8.91}$ c) $\frac{7.801}{23.015}$

3. a) $\frac{1.6}{7}$ b) $\frac{\sqrt{2}}{\sqrt{6}}$

4. Write each of these as percents.

   a) 0.19 b) 0.015 c) 2.15 d) 3

5. Write each of these as fractions in simplest form.

   a) 10% b) 25% c) 62.5% d) 137.5%

6. Calculate each of the following.

   a) 70% b) 120% of 84 c) 38% of 751

7. Calculate each of the following.

   a) $7\frac{1}{2}\%$ of $20,000 b) .02% of 27,481 c) 1.05% of 845

8. Compute each of these mentally.

   a) 50% of 840 b) 10% of 2480
   c) 12.5% of 48 d) 125% of 24
   e) 200% of 56 f) 110% of 180

9. Mentally convert each of these to a percent.

   a) $\frac{7}{28}$ b) $\frac{11}{33}$ c) $\frac{72}{144}$ d) $\frac{44}{66}$

10. Mentally estimate the number that should go in the blank to make each of these true.

    a) 27% of ________ equals 16.
    b) 4 is ________ % of 7.5.
    c) 41% of 120 = ________.

Percent

Activity 2

1. Arrange the following in order of increasing size.

$$\frac{19}{25},\quad 0.\overline{7},\quad 77\%,\quad \text{and} \quad \frac{15}{19}$$

2. In a given population of men women, 40% of men are married and 30% of the women are married. What percentage of the adult population is married?

3. Show that the sale price of items marked down 15% is the same as 85% of the retail price.

4. During the first half of a basketball game, the basketball team at a High School made 60% of their 40 field goal attempts. During the second half, they scored on only 25% of 44 attempts from the field. To the nearest 1%, what was their field goal shooting percentage for the entire game?

5. When asked about his performance in an upset victory in a football game, the quarterback said that he gave 110% effort. Briefly discuss the reasonableness of this assertion.

Theme II: Real Numbers: Decimals and Percents

Lesson 5: Placing Numbers in Perspective – Scientific Notation

Do You Know?

What is a billion dollars or how much is a trillion dollars? For many people, these numbers are just words – so large that they hardly seem to mean anything. But we hear numbers like these every day, and we cannot truly understand the major issues of our time unless we have some understanding of the numbers that are involved. In this unit, we will study several techniques for putting large or small numbers into a perspective that gives them real meaning.

Writing Large and Small Numbers

Working with large and small numbers is much easier when we write them in a special format known as **scientific notation**. We express numbers in this format by writing a number *between* 1 and 10 multiplied by a power if 10. For example, a billion is ten to the ninth power, or 10^9, so we write 6 billion in scientific notation as $6 \text{ x } 10^9$. Similarly, we write 420 in scientific notation as 4.2×10^2, and 0.67 as 6.7×10^{-1}.

Scientific Notation

Scientific Notation is a format in which a number is expressed as a number *between* 1 and 10 multiplied by a power of 10. Scientific notation makes it easy to write numbers no matter how large or small they might be. We must be careful, however, not to let this ease of writing deceive us. For example, it is easy to write the number 10^{80} that we might think it is not all that big – but, in fact it is a number larger than the total number of atoms in the known universe.

EXAMPLE 1 Numbers in Scientific Notation
Rewrite each of the following statements using scientific notation.

a. Let's say the World Bank has about $9,100,000,000,000.
b. The diameter of a hydrogen nucleus is about 0.000000000000001 meter.

Solution Notice how much easier it is to read the numbers with scientific notation.

a. The World Bank has about $\$9.1 \times 10^{12}$, or $9.1 trillion.
b. The diameter of a hydrogen nucleus is about 1×10^{-15} meter.

Placing Numbers in Perspective – Scientific Notation

Approximations with Scientific Notation
Another advantage of scientific notation is that it makes it easy to approximate answers without a calculator. For example, we can quickly approximate the answers to 5795 × 326 by rounding 5795 to 6000 and 326 to 300. Writing the rounded numbers in scientific notation, we then see that

5795 × 326 is approximately $(6 \times 10^3) \times (3 \times 10^2) = 18 \times 10^5 = 1{,}800{,}000$
Because the exact answer is 1,889,170, this approximation provides a good estimate.

EXAMPLE 2 Checking Answers with Approximations
You and a friend are doing a rough calculation of how much garbage new City residents produce every day. You estimate that, on average, each of the 8 million residents produce 1.8 pounds, or 0.0009 ton, of garbage each day. Thus, the total amount of garbage is

$$8{,}000{,}000 \text{ persons} \times 0.0009 \frac{\text{ton}}{\text{person}}$$

Your friend quickly presses calculator buttons and tells you that the answer is 225 tons. Without using your calculator, determine whether this answer is reasonable.

Solution You can write 8 million as 8×10^6, which is nearly 10. You can write 0.0009 as $9 \times$X 10^{-3}, which is nearly 10. Thus, the product should be approximately

$$10^7 \text{ X } 10^{-3} = 10^{7-3} = 10^4 = 10{,}000$$

Clearly, your friend's answer of 225 tons is too small. This simple approximation technique provides a useful check, even though it did not tell us the exact answer.

Scientific Notation - Operations
To convert a number from ordinary notation to scientific notation:
Step 1. Move the decimal point to come after the *first* nonzero digit.
Step 2. For the power of 10, use the number of places the decimal point moves; the power is *positive* if the decimal point moves to the left and *negative* if it moves to the right.

Examples:

3042 → $3.042 \text{ X } 10^3$ Decimal moves 3 places to the left.

0.00012 → $1.2 \text{ X } 10^{-4}$ Decimal moves 4 places to the right.

$226 \text{ X } 10^2$ → $(2.26 \text{ X } 10^{2)} \text{ X } 10^2 = 2.26 \text{ X } 10^4$ Decimal moves 2 places to the left.

Placing Numbers in Perspective – Scientific Notation

Converting from Scientific Notation
To convert a number from scientific notation to ordinary notation:
Step 1 The power of 10 indicates how many places to move the decimal point; move it to the right if the power of 10 is positive and to the left if it is negative.
Step 2 If moving the decimal point creates any open places, fill them with zeros.

Examples:
$4.01 \times 10^2 \rightarrow 401$ Move the decimal 2 places to the right.

$3.6 \times 10^6 \rightarrow 3{,}600{,}000$ Move the decimal 6 places to the right.

$5.7 \times 10^{-3} \rightarrow 0.0057$ Move the decimal 3 places to the left.

Multiplying or Dividing with Scientific Notation
Multiplying or dividing numbers in scientific notation simply requires operating on the powers of 10 and the other parts of the number separately.

Examples:

$$(6 \times 10^2) \times (4 \times 10^5) = (6 \times 4) \text{ X } (10^2 \times 10^5)$$
$$= 24 \times 10^7$$
$$= 2.4 \times 10^8$$

$$\frac{4.2 \times 10^{-2}}{8.4 \times 10^{-5}} = \frac{4.2}{8.4} \times \frac{10^{-2}}{10^{-5}}$$
$$= 0.5 \times 10^{-2-(-5)}$$
$$= 0.5 \times 10^3$$
$$= 5 \times 10^2$$

Note that, in both examples, we first found an answer in which the number multiplied by a power of 10 was *not* between 1 and 10. We the followed the process for converting the final answer into scientific notation.

Addition and Subtraction with Scientific Notation
In general, we must write numbers in ordinary notation before adding or subtracting.

Examples:

$$(3 \times 10^6) + (5 \times 10^2) = 3{,}000{,}000 + 500$$
$$= 3{,}000{,}500$$
$$= 3.0005 \times 10^6$$

$$(4.6 \times 10^9) - (5 \times 10^8) = 4{,}600{,}000{,}000 - 500{,}000{,}000$$
$$= 4{,}100{,}000{,}000$$
$$= 4.1 \times 10^9$$

Placing Numbers in Perspective – Scientific Notation

We know both numbers have the *same* power of 10, we can factor out the power of 10 first.

Examples:

$$(7 \times 10^{10}) + (4 \times 10^{10}) = (7 + 4) \times 10^{10}$$
$$= 11 \times 10^{10}$$
$$= 1.1 \times 10^{11}$$

$$(2.3 \times 10^{-22}) - (1.6 \times 10^{-22}) = (2.3 - 1.6) \times 10^{-22}$$
$$= 0.7 \times 10^{-22}$$
$$= 0.7 \times 10^{-23}$$

Placing Numbers in Perspective Through Comparisons

A second general way to put numbers in perspective is by making comparisons. For example, consider \$100 billion, which is roughly the wealth of the world's richest individuals. It's easy to say a number like 100 billion, but how big is it? Let us think of it in terms of counting. Suppose you were asked to count \$100 billion in \$1 bills. How long would it take? Clearly, if we assume you can count 1 Bill each second, it would take 100 billion seconds. We can put 100 billion (10^{11}) seconds in perspective by converting to years, using a chain of conversions:

We see that 100 billion seconds is equivalent to 3171 years. In other words, you would need more then *three thousand years* just to count \$100 billion in \$1 bills. And that assumes that you never take a break: no sleeping, no eating, and absolutely no dying!

Putting Numbers in Perspective - Scientific Notation

Activity 1

1. Convert each of the following numbers from scientific to ordinary notation and write its name. *Example:* $2 \times 10^3 = 2000$ = two thousand
   a. 3×10^3 b. 6×10^6 c. 3.4×10^5 d. 2×10^{-2}
   e. 2.1×10^{-4} f. 4×10^{-5}
2. Convert each of the following numbers from scientific to ordinary notation and write its name.
   a. 8×10^2 b. 5×10^3 c. 9.6×10^4 d. 2×10^{-3}
   e. 3.3×10^{-5} f. 7.66×10^{-2}
3. Write each of the following numbers in scientific notation.
   a. 233 b. 123,547 c. 0.11 d. 9736.23
   e. 124.58 f. 0.8642
4. Write each of the following numbers in scientific notation.
   a. 4327 b. 984.35 c. 0.0045 d. 624.87
   e. 0.1357 f. 98.180004
5. Do the following operations and show your work clearly. Be sure to express answers in scientific notation.
   You may round your answers to one decimal place (as in 3.2×10^5).
   a. $(3 \times 10^3) \times (2 \times 10^2)$ b. $(4 \times 10^2) \times (3 \times 10^8)$
   c. $(3 \times 10^3) + (2 \times 10^2)$ d. $(8 \times 10^{12}) \div (4 \times 10^4)$
6. Do the following operations and show your work clearly. Be sure to express answers in scientific notation.
   You may round your answers to one decimal place (as in 3.2×10^5).
   a. $(4 \times 10^7) \times (2 \times 10^8)$ b. $(3.2 \times 10^5) \times (2 \times 10^4)$
   c. $(4 \times 10^3) + (5 \times 10^2)$ d. $(9 \times 10^{13}) \div (3 \times 10^{10})$

In problems 7 and 8, compare each pair of numbers. By what factor do the numbers differ?

7. a. 10^{35}, 10^{26} b. 10^{17}, 10^{27} c. 1 billion, 1 million
   d. 7 trillion, 7 thousand e. 2×10^{-6}, 2×10^{-9} f. 6.1×10^{27}, 6.1×10^{29}

8. a. 250 million, 5 billion b. 9.3×10^2, 3.1×10^{-2} c. 10^{-8}, 2×10^{-13}
   d. 3.5×10^{-2}, 7×10^{-8} e. 1 thousand, 1 thousandth f. 10^{12}, 10^{-9}

9. The diameter of a typical bacterium is about 0.000001 meter.

10. A beam of light can travel the length of a soccer field in about 30 nanoseconds. Express your answer in seconds. (*Hint: Recall that nano means one billionth*.)

Putting Numbers in Perspective - Scientific Notation

Activity 2

In Problems 1-2 make an estimate of the answer without a calculator, showing your method of estimation. Then do the exact calculation (with a calculator if necessary), and describe how well your approximation technique worked.

1. a. $300,000 \times 100$ b. 5.1 million × 1.9 thousand
   c. $4 \times 10^9 \div 2.1 \times 10^6$ d. 33 million × 3.1 thousand
   e. $4,288,364 \div 2132$ f. $(6.129845 \times 10^6) \div (2.198 \times 10^4)$

2. a. 5.6 billion ÷ 200 b. 4 trillion ÷ 260 million
   c. $9000 \times 54,986$ d. 3 billion × 25,000
   e. $5987 \div 341$ f. $43 \div 765$

Decide whether each of the following statements makes sense (or is clearly true) or does not make sense (or is clearly false). Explain your reasoning.

3. I read a book that had 10^5 words in it.

4. I've seen about 10^{50} commercials on TV.

5. During a recent sold-out soccer game, at a large stadium, the star player signed autographs for every single person in attendance.

Theme: III
Organizing and Representing Numbers and Data

Theme III: Organizing and Representing Numbers and Data

Lesson 1: Sets and Their Operations

Do You Know?

A **set** is a well-defined collection of objects where the arrangement of the elements in the collection is not important. Sets are usually denoted by capital letters: A, B, C, …J, …X, Y,. Small or lower case letters a, b, c, …, x, y and numbers are typically used to denote the **elements** or members of a set.

Example: B = $\{l, m, n, o, p, q\}$. Two typical ways to denote that p is a member of the set B are

1) "P is an element of B;"
2) P $\in$ B

Similarly, one may state that

3) "a is not an element of B"; or
4) a $\notin$ B

One may denote a set by describing its elements; example A = {x : x is a real number and x is not a rational}. That is A is the set of all irrational numbers; numbers such as $\sqrt{2}$, 3$\sqrt{5}$, etc.

Subsets
If every element in a set B is also an element in set A, then B is called a **subset** of A. Symbolically we denote use the notation B $\subset$ A, meaning B is strictly a subset of A and B is smaller than A; or B $\subseteq$ A, meaning B is a subset of A and B may be equal to A.

Two sets C and D are **equal** if each set has the exact same elements. Example:

If C = {1, 2, 3, 4} and D = {1, 3, 4, 3} then C and D are equal. This also means that C $\subseteq$ D and D $\subseteq$ C.

However, the two sets A = {x, y, z} and B {1, 2, 3} are not equal. The set with no elements is called the **empty set**, and it is denoted by Ø. Ø = {}.

Sets and Their Operations

Every set has two (2) obvious subsets;

$A \subseteq A$ and $ø \subseteq A$.

Also, if $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.

For example, consider A = {x, z}, B = {x, y, z} and C = {w, x, y, z}
The number of elements of a set A is called the **cardinality** of A and this is denoted symbolically by n (A).

Example: If A = {x, y, z} then n (A) = 3
If n (A) = n, for some natural number n, A is said to be **finite**; otherwise A is said to have an **infinite** number of elements. A = {x, y, z} is finite. B = {set of all even … numbers is infinite.

Universal Set

In any application of sets, usually there are many different sets under consideration. Moreover, we often place all sets under consideration in some fixed large set call the **Universal set**, denoted by U.

Examples:

If for a given application, all the sets under consideration are
A = {1, 2, 3, 4, 5,}
B = {x; x is an integer, x = 2n for some integers n}
C = {1, 3, 5, …, 2n – 1, …}.
D = {1, -2, -3. -4, -5, -6, -7}

Then for these sets the universal set could be selected as Z, the set of all integers, i.e. U = Z.

If we were considering sets of different people from different countries from various continents, then the universal set might be selected as U = {the world population of all persons}.

The Power set P(A) of Set A

We define the **power set** of set A as the set of all subsets of A and it is denoted by P(A).
Example if A = {a, b, c, d}, then P(A) = {{a, b, c, d}, {a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}, {a, b}, {a, c,}, {a, d}, {b, c}, {b, d}, {c, d}, {a}, {b}, {c}, {d}, ø}

It is not an accident that n(A) = 4 and n(P(A)) = 16.
What is 2^4? Hint: $(P(A)) = 2^{n(A)}$

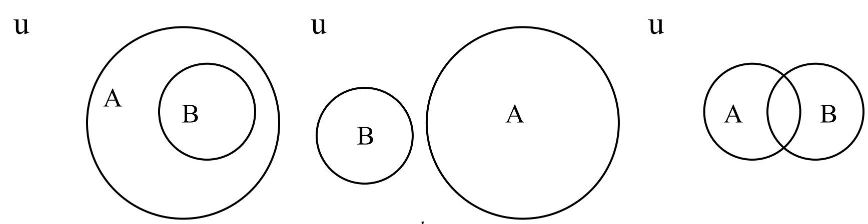
Sets and Their Operations

Venn Diagrams

A Venn diagram is a pictorial representation of sets in which sets are represented by circled areas in the plane.

The universal set U is represented by the interior of a rectangle, and the other sets are represented by circular figures lying within the rectangle. If $A \subseteq B$, then the circular figures representing A will be entirely within the circular figures representing B as in Figure I(a). if A and B are **disjoint**, i.e., if they have no elements in common, then the circular figures representing A will be separated from the circular figures representing b as in Figure I(b).

However, if A and B are two arbitrary sets, it is possible that some objects are in A but not in B, some are in B but not in A, some are in both A and B, and some are in neither A nor B; hence in general we represent A and B as in Figure H© .



a *b* *c*

Figure I

Partitions

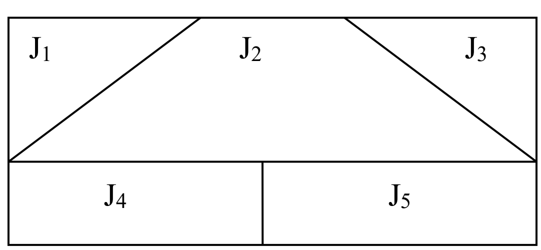
Let J be a nonempty set. A partition of J is a subdivision of J into nonoverlapping, nonempty subsets. Precisely, a partition of J is a collection $\{J_i\}$ of nonempty subsets of J such that:

(i) Each a in J belongs to one of the susets J_i

(ii) The sets of $\{J_i\}$ are mutually disjoint; that is, if

$J_i \neq J_k$ then $A_i \cap A_k = ø$

The subsets in a partition are called **cells**. Figure II is a Venn diagram of a partition of the rectangular set J of points into five cells, J_1, J_2, J_3, J_4, and J_5.



Sets and Their Operations

Union, Intersection, Complement, and Difference

The **union** of two sets a and B, denoted by $A \cup B$, is the set of all elements which belong to A or to B; that is,

$A \cup B = \{x: x \in A \text{ or } x \in B\}$

Here "or" is used in the sense of and/or.
Can you illustrate this with a Venn Diagram?

The **intersection** of two sets A and B, denoted by $A \cap B$, is the set of elements which belong to both A and B; that is,

$A \cap B = \{x: x \in A \text{ and } x \in B\}$
Can you illustrate this with a Venn Diagram?

If $A \cap B = ø$, that is, if A and B do not have any elements in common, then A and B are said to be disjoint or nonintersecting.

Can you do the following?

Let A = {1, 2, 3, 4}, b = {3, 4, 5, 6, 7}, and C {2, 3, 5, 7}. Find (a) $A \cup B$; (b) $A \cap B$; (c) $A \cup C$; (d) $A \cap C$.

The operation of set inclusion is closely related to the operations of union and intersection, as shown by the following theorem.

Theorem A: The following are equivalent: $A \subseteq B$, $A \cap B = A$, and $A \cup B = B$.

Sets and Their Operations

Complements and Differences

Recall that all sets under consideration at a particular time are subsets of a fixed universal set U. The **complement** of a set A, denoted by A^c, is the set of elements which belong to U but which do not belong to A; that is

$$A^c = \{x: x \in U, x \notin A\}$$

The difference of A and b, denoted by A - B, is set the set of elements which belong to A but which do not belong to B; that is

$A - B = \{x: x \in A, x \notin B\}$
The set A\B is read "A minus B".

The **symmetric difference** of sets A and B, denoted by $A \oplus B$, consists of those elements which belong to A or B but not to both; that is,

$$A \oplus B = (A \cup B) - (A \cap B)$$

One can also show that

$$A \oplus B = (A - B) \cup (B - A)$$

For example, suppose A = {1, 2, 3, 4, 5, 6} and B = {4, 5, 6, 7, 8, 9}. Then, A - B = {1, 2, 3}, B - A = {7, 8, 9} and so $A \oplus B$ = {1, 2, 3, 7, 8, 9}

Laws for the Algebra of Sets

Idempotent laws	
(1a) $A \cup A = A$	(1b) $A \cap A = A$
Associative laws	
(2a) $(A \cup B) \cup C = A \cup (B \cup C)$	(2b) $(A \cap B) \cap C = A \cap (B \cap C)$
Commutative laws	
(3a) $A \cup B = B \cup A$	(3b) $A \cap B = B \cap A$
Distributive laws	
(4a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	(4b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Identity laws	
(5a) $A \cup ø = A$ (6a) $A \cup U = U$	(5b) $A \cap U = A$ (6b) $A \cap ø = ø$
Involution laws	
(7) $(A^c)^c = A$	
Complement laws	
(8a) $A \cup U^c = U$ (9a) $U^c = ø$	(8b) $A \cap A^c = ø$ (9b) $ø^c = U$
DeMorgan's laws	
(10a) $(A \cup B)^c = A^c \cap B^c$	(10b) $(A \cap B)^c = A^c \cup B^c$

Sets and Their Operations

Activity I

1. Give a verbal description of each of the following sets:
   a. A = {a, e, i, o, u, y} b. B = {1, 2, 3, 4, 5, 6, 7, 8, 9}
   c. {Sunday, Monday, Tuesday, Wednesday, Thursday, Friday, Saturday}

2. Give a verbal description of each of the following sets:
   a. {…, -5, -3, -1, 1, 3, 5, …}
   b. {Mercury, Venus, Earth, Mars, Jupiter, Saturn, Uranius, Neptune, Pluto}
   c. {January, February, March, …, October, November, December}

3. Write in set notation as a listing, the following sets:
   a. The set of neutral or counting numbers b. The set of whole numbers
   c. The set of all integers

4. Write a set notation describing the following sets:
   a. The set of rational numbers
   b. The set of irrational numbers
   c. The set of real numbers

5. Describe each of the sets using the listing method.
   a. {x: x is an odd counting or natural number with one digit}
   b. {x: $x^2 = 9$}
   c. {x: x is an even integer and x > 0}

6. Which of the following sets are equal.
   A = {1, 2, 3, 4, 5, 6}; B. {a, b, c, d, e}; C = {1, 4, 2, 5, 3}
   D = {x: x is the first five letters of the alphabet}
   E = {x: x is a natural number less than 5}

7. Let A = {x, y, z}
   a. What is the cardinality of A, i.e. n(A) = ?
   b. What is the power set of A, that is list the members of P(A)
   c. Name two subsets that exist for all sets

8. Let A = {1, 2, 4, 4}, B= {3, 4, 5, 6, 7}; and C = {2, 3, 5, 7} and let U = {x: x is less than or equal 20. What is
   a. $A \cup B$; b. $A \cap B$; c. $A \cup C\ C^c$

9. Use Venn diagrams to illustrate all problems and answers in problem 8.

10. Give two examples of finite sets and two examples of infinite sets.

Sets and Their Operations

Activity II

1. Given A = {1, 2, 3, 4,}, B = {3, 4, 5, 6, 7}, C = {2, 3, 5, 7}
   a. What is A – C? b. What is A ⊕ B = ?

2. Use the sets A, B, and C in problem 1 to illustrate the communitative laws of union and intersections.

3. Use the sets A, B, and C in (1) to illustrate the distributive laws

4. Illustrate how: (a) Natural numbers, (b) whole numbers, (c) Integers

   (d) Rational numbers, and (e) real numbers relate to one another using Venn Diagram

5. Given the set A = {x, y, z}, display P(A)? Illustrate at least three (3) ways to partition the set P(A).

6. - 10.
   Create and find five (5) problems about set operations and solve them.

Theme III: Organizing and Representing Numbers and Data

Lesson 2: Relations

Do You Know?

Product Sets – Cartesian Product

Consider two arbitrary sets A and B. The set of all ordered pairs (a, b) where a $\in$ A and b $\in$ B is called the **product set**, or **Cartesian product**, of A and B. A short way to write this product is A x B, which is read "A cross B". That is

A x B = {(a,b): a $\in$ A and b $\in$ B.

Example If A = {1, 2} and B = {a, b, c}. Then

A x B = {(1,a), (1,b), (1,c), (2,a), (2,b), (2,c)}

Also

A x A = {(1,1), (1,2), (2,1), (2,2)}

There are two things worth stating about these examples. First of all, A x B $\neq$ B x A. The Cartesian products deal with ordered pairs, so naturally the order in which the elements are considered is important. Secondly, using n(S) for the number of elements in a set S, we have

n(A x B) = 6 = 2 · 3 = n(A) · n(B)

In fact, n(A x B) = n(A) · n(B) for any finite sets A and B. This follows from the observation that, for an ordered pair (a, b) in A x B, there are n(A) possibilities for a, and for each of these, there are n(B) possibilities for b.

The idea of a product of sets can be extended to any finite number of sets. For any sets $A_1,\ldots,A_n$ and is denoted by

$$A_1 \text{ x } A_2 \text{ x} \ldots \text{x } A_n \text{ or } \prod_{i=1}^{n} A_i$$

Relations

Relations

Let A and B be two arbitrary sets. A **binary relation** or simply, **relation** from A to B is a subset of A x B.

Thus if R is a relation from A to B. Then R is a set of ordered pairs. i.e.

$R = \{(a, b)$ where $a \in A$ and $b \in B\}$.

Theorem A

If R is a relation from a set A to a set B, then for every $(a, b) \in A \times B$

(i) $(a,b) \in R$; we then say "a is R-related to b" and write aRb; or

(ii) $(a,b) \notin R$; we then say "a is not R-related to b" and write a/b.

If R is a relation from a set A to itself, that is R is a subset of $A^2 = A \times A$, then we say that R is a **relation on A**.

The **domain** of a relation R is the set of all first elements of the ordered pairs that belong to R, and the **range** of R is the set of all second elements.

Inverse Relation

Let R be any relation from a set A to a set B. The inverse of R, denoted by R^{-1} , is the relation from B to A which consists of those ordered pairs which, when reversed, belong to R; that is = $R^{-1}=\{(b,a): (a,b) \in R\}$

For example, the inverse of the relation R = {(1,y), (1,z), (3,y)} from A = {1, 2, 3} to B = {x, y, z} is given by:

$$R^{-1}=\{(y,1), (z,1),(y,3)\}$$

Clearly, if R is any relation, then $(R^{-1})^{-1} = R$. Also, the domain and range of R^{-1} are equal, respectively, to the range and domain of R. Moreover, if R is a relation on A, then R^{-1} is also a relation on A.

Some Pictoral Representations of Relations

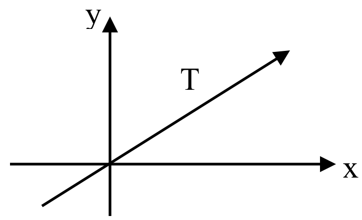
First, we consider a relation T on the set **R** of real numbers; that is T is a subset of $\mathbf{R^2 = R \times R}$. Since $\mathbf{R^2}$ can be represented by the set of points in the plane, we can represent T by those points in the plane that belong to T. This pictorial representation of the relation is sometimes called the **graph** of the relation T. Another way of viewing the relation T as all ordered pairs of real numbers which satisfy some given equation

$$E(x,y)=0$$

In this case, the graph of the relation T is the same as the graph of the equation it satisfies.

Example
Let T on R x R be

T: {(x,x): x ∈ R and x = x} Then T = {…(-1,-1), (-½, -½), (0,0), (½,½), (1,1),…}



Composition of Relations

Let A, B and C be sets, and let R be a relation from A to B and let S be a relation from B to C. That is, R is a subset of A x B and S is subset of B x C. then R and S can define a relation from A to C denoted by R ∘ S = {(a,c): where there exists b ∈ B for which (a,b) ∈ R and (b,c) ∈ S}

The relation R ∘ S is called the **composition of R and S**; it is sometimes denoted simply by RS.

Suppose R is a relation on a set A, that is R is a relation from a set A to itself. Then R ∘ R is sometimes denoted by R^2. Similarly, $R^3 = R^2 \circ R = R \circ R \circ R$. Similarly, R^n is defined for all positive n.

Special Relations on a Set A.

Consider a given set A. There are a number of important types of relations that are defined on A.

Reflective Relation

A relation R on a set A is **reflective** if aRa for every a ∈ A, that is, (a, a) ∈ R for every a ∈ A. Thus, R is not reflective if there exists an a ∈ A such that (a, a) ∉ R.

Relations

Symmetric and Antisymmetric Relations

A relation R on a set A is **symmetric** if whenever aRb then bRa, that is, if whenever (a, b) $\in$ R then (b, a) $\in$R. Thus, R is not symmetric if there exists a, b$\in$ A such that (a, b) $\in$R but (b, a) $\notin$ R.

A relation R on a set A is **antisymmetric** if whenever aRb and bRa, then a =b, that is, if whenever (a, b), (b, a) $\in$ R then a = b. Thus, R is not antisymmetric if there exist a, b $\in$ R such that (a, b) and (b, a) belong to R, but a $\neq$ b.

Transitive Relations

A relation R on a set A is **transitive** if whenever aRb and bRc then aRc, that is, if whenever (a, b), (b, c) $\in$ R then (a, c) $\in$ R. Thus, R is not transitive if there exist a, b, c $\in$ A such that (a, b), (b, c) $\in$R but (a, c) $\notin$ R.

Examples: Which relation is reflective, symmetric, transitive?

Given the relation R on the set A, determine whether R is reflective, symmetric, antisymmetric or transitive; A = {1, 2, 3} amd R = {(1, 2), (1, 2), (1, 2), (2, 1), (2, 2), (3, 1), (3, 3)}.

Equivalence Relations

Consider a nonempty set S. A relation R on S is an **equivalence relation** if R is reflevive, symmetric, and transitive. That is, R is an equivalence relation on S if it has the following three properties:

1. For every a $\in$ S, aRa
2. If aRb, then bRa
3. If aRb and bRc, then aRc.

The general idea behind an equivalence relation is that it is a classification of objects that are in some way “alike”. In fact, the relation “=” of equality on any set S is an equivalence relation; that is:

1. a = a for every a $\in$ S.
2. If a = b, then b = a
3. If a = b and b = c, then a = c.

Relations

Activity 1

1. Let A = {1, 2, 3}; B = {2, 4, 6, 8} and C = {8, 9}
   a. How many elements are in A x B; List them
   b. How many elements are in B x A; List them
   c. Is A x B = B x A?

2. Using sets A and B in problem 1,
   a. How many elements are in A x A? List them
   b. How many elements are in C x C? List them

3. Using the set B in problem 1, List the following relations on B as a set of ordered pairs:
   a. R1 where aRb means a is greater than or equal to b
   b. R2 where aRb means a is less than b.

4. Given a set A = {1, 2, 3, 4} list an anti-symmetric relation on A.

5. Determine whether each of the following relations is reflexive, symmetric or transitive on the given set A.

   a. R1 = {(1, 2), (1, 2), (2, 1), (2, 2), (3, 3)} on A = {1, 2, 3}
   b. R2 = {(a, a), (a, b), (b, b), (b, c), (c, b), (c, c)} on A = {a, b, c}
   c. R3 = {(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (3, 1), (3, 3), (4, 4)} on A = {1, 2, 3, 4}

6. In problem 5 is R1, R2 or R3 an equivalence relation?

7. In problem 5 what ordered pairs would you add to each relation(s), that are not equivalence, that would make them equivalent relation(s)?

8. In problem 5 what is the domain set of R1, R2, R3?
   In problem 5 what is the range set of R1, R2, R3?

9. Let A = R, the set of reals, determine whether each of the following relations is reflexive, symmetric, antisymmetric, transitive?

   (a) R1 aRb means that a is less than b;
   (b) R2 aRb means that a = b;
   (c) R3 aRb means that a is not equal to be.
   (d) R4 aRb means that a is not equal to b.

10. In problem 9, which of the relations is an equivalent relation?

Relations

Activity 2

1. Consider the set A = {1, 2, 3, 4, 5, 6}.
   a. List the elements of the relation R1 on A where aRb means, "a divides b."
   b. List the elements of the relation R2 on A where aRb means, "a is less than or equal b."

2. Consider the relations R1 and R2 in problem 1,
   a. What is $R1^{-1}$? List ordered pairs.
   b. What is $R2^{-1}$? List ordered pairs.

3. Consider the relations R1 and R2 in problem 1,
   a. What is $R1 \cap R2$? List
   b. What is $R1 \cup R2$? List

4. Consider the set A = {1, 2, 3, 4} and the relation R = {(1, 2), (1, 3), (2, 3), (2, 4), (3, 1)} and the relation S = {(2, 1), (3, 1), (3, 2), (4, 2)}. Write $S \circ R$ as a set of ordered pairs.

5. Give an example of a relation on the set A = {1, 2, 3, 4} such that
   a. R1 is both symmetric and antisymmetric.
   b. R2 is neither symmetric nor antisymmetric.

Theme III: Organizing and Representing Numbers and Data

Lesson 3: Functions

Do You Know?

A **function** from a set A to a set B, is a relation from A to B in which **each element** in A is assigned exactly one element in B. The elements in A are called the **domain** of the function. The elements in B to which the elements in A are assigned is called the **images** or **codomain** of the elements of A or the **range** of the function. Functions are usually denotes by lowercase letters like f, g, h, …..q, ….. For example:

$$\begin{aligned} f: A &\longrightarrow B \\ a &\longrightarrow f(a) \end{aligned}$$

denotes that "f is a function from A to B" where 'a is assigned to f(a) , an element of B."

Example

Which of the following relations from A to B are functions from A to B?

A = {1, 2, 3}   B = {1, 2, 3, 4, 5}

(a) {(1, 2), (2, 3), (3, 5)};   (b) {(1, 1), (2, 1), (3, 1)}

(c) {(1, 3), (1, 5), (3, 5)};   (d) {(1, 2), (2, 4)}

Functions may be expressed in different ways, as relations are expressed in different ways. Specifically, whenever a function, say f, is given by a formula in terms of a variable *x*, it is usually assumed that the domain of the functions is R, the set of real numbers, and that the images of *f*, the range of *f*, are elements of R.

Example

Let *y* be a function on R, the set of real numbers, and we define $g(x) = x^2 - 1$, find g(2), g(0), g(1), and g(a).

Solution

$g(2) = 2^2 - 1 = 4 - 1 = 3$; $g(0) = 0^2 - 1 = -1$; $g(-1) = (-1)^2 - 1 = 1 - 1 = 0$;
$g(a) = a^2 - 1 = g(a + 1) = (a + 1)^2 - 1 = (a^2 - 2a + 1) - 1 = a^2 + 2a$

Functions

Remark: There is an important connection shared by functions and relations: Every function is a relation but not every relation is a function. Thus, functions have all the properties that relations have.

One-to-One, Onto, and Invertible Functions

A function f: A $\longrightarrow$ B is said to be **one-to-one** (also written 1-1) if different elements in the domain A have distinct images. Another way of saying the same thing is that f is **one-to-one** if *f(a) = f(a')* implies *a = a'*.

A function f: A $\longrightarrow$ B is said to be an **onto function** if each element of B is the image of some element of A. In other words, f: A $\longrightarrow$ B is onto if the image of *f* is the entire codomain (example*,* if *f*(A) = B). In such a case, we say that *f* is a function from A onto B or that *f* maps A onto B.

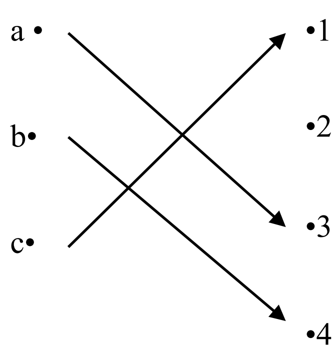
A function f: A $\longrightarrow$ B is **invertible** if there is a function g: B $\longrightarrow$ A such that g{f(a)} = u and g{k(a)} = a. In general, there may not be such a function. But if there is one, then it is unique, it is denoted by f^{-1}, and *f* is said to be invertible. The following theorem gives simple criteria for invertibility.

Theorem A: A function f: A $\longrightarrow$ B is invertible if and only if *f* is both one-to-one and onto.

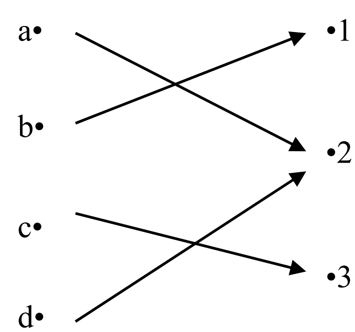
If f: A $\longrightarrow$ B is one-to-one on onto, then *f* is called a **one-to-one correspondence** between A and B. This terminology comes from the fact that each element of A will then correspond to a unique element of B and vice versa. f^{-1}simply reverses the direction of this correspondence.

Different Types of Functions

(a) One-to-one, not onto

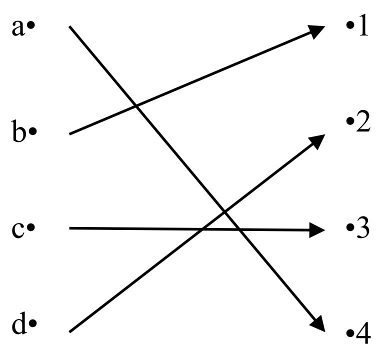


(b) Onto, not one-to-one

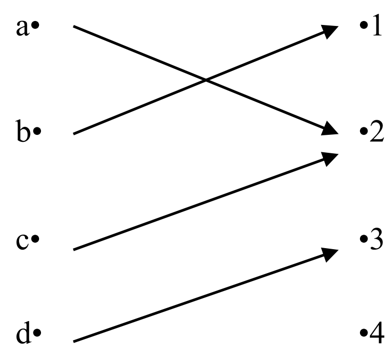


Functions

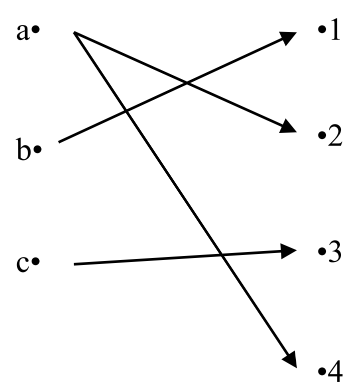
(c) One-to-one, and onto



(d) Neither one-to-one nor onto



(d) Not a function



It might be interesting to include in this lesson some special mathematical functions.

Floor and Ceiling Functions

Let x be any real number. Then x lies between two integers called the floor and the ceiling of x. Specifically,

> $\lfloor x \rfloor$, called the **floor** of x, denotes the greatest integer that does not exceed x.
> $\lceil x \rceil$, called the **ceiling** of x, denotes the least integer that is not less than x.

If x is itself an integer, then $\lfloor x \rfloor = \lceil x \rceil$, otherwise $\lfloor x \rfloor + 1 = \lceil x \rceil$.

Functions

Integer and Absolute Value Functions

Let x be any real number. The **integer value** of x, written INT(x), converts x into an integer by deleting (truncating) the fractional part of the number. Thus,

$$\text{INT}(3.14) = 3, \quad \text{INT}(\sqrt{5}) = 2, \quad \text{INT}(-8.5) = -8$$

The **absolute value** of the real number x, written ABS(x) or $|x|$, is defined as the greater of x or $-x$. Hence ABS(0) = 0, and for $x \neq 0$, ABS(x) = x or ABS(x) = $-x$, depending on whether x is positive or negative. Thus,

$$|-15| = 15, \qquad |7| = 7, \qquad |-3.33| = 3.33$$

We note that if the $|x| = x$ and, for $x \neq 0$, then $|x|$ is positive.

Composition of Functions

Let g be a function from the set A to the set B and let f be a function from the set B to the set C. The **composition of functions f and g**, denoted by $f \circ g$, is defined by

$(f \circ g)(a) = f(g(a))$.

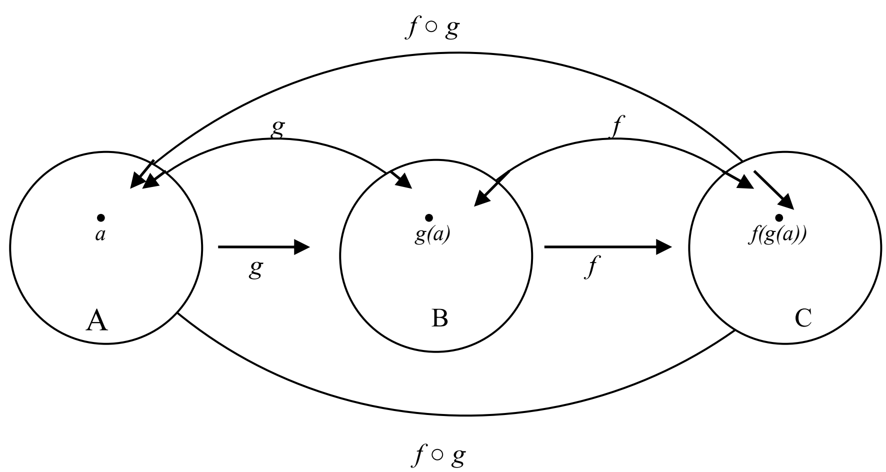


Figure I The composition of the Functions f and g.

Functions

Example

Let g be the function from the set $\{a, b, c\}$ to itself such that $g(a) = b$, $g(b) = c$, and $g(c) = a$. let f be the function from the set $\{a, b, c\}$ to the set $\{1, 2, 3\}$ such that $f(a) = 3$, $f(b) = 2$, and $f(c) = 1$. What is the composition of f and g, and what is the composition of g and f?

Solution

The composition of $f \circ g$ is defined by $(f \circ g)(a) = f(g(a)) = f(b) = 2$, $(f \circ g)(b) = f(g(b) = f(c) = 1$, and $(f \circ g)(c) = f(g(c)) = f(a) = 3$.

Note that $g \circ f$ *is not defined, because the range of* f *is not a subset of the domain of* g.

Example

Let f and g be the functions from the set of integers to the set of integers defined by $f(x) = 2x + 3$ and $g(x) = 3x + 2$. What is the composition of f and g? what is the composition of g and f?

Solution

Both the compositions $f \circ g$ and $g \circ f$ are defined. Moreover,

$$(f \circ g)(x) = f(g(x) = f(3x + 2) = 2(3x + 2) + 3 = 6x + 7$$

and

$$(g \circ f)(x) = g(f(x)) = g(2x + 3) = 3(2x + 3) + 2 = 6x + 11$$

Remark Note that even though $f \circ g$ and $g \circ f$ are defined for the functions f and g in the example, $f \circ g$ and $g \circ f$ are not equal. In other words, the commutative law does not hold for the composition of functions.

The Graphs of Functions

We can associate a set of pairs in A × B to each function from A to B. This set of pairs is called the **graph** of the function and is often displayed pictorially to aid in understanding the behavior of the function.

Definition

Let f be a function form the set A to the set B. The **graph** of the function f is the set of ordered pairs $[(a, b) \mid a € A \text{ and } f(a) = b\}$.
From the definition, the graph of a function f from A to B is the subset of A × B containing the ordered pairs with the second entry equal to the element of B assigned by f to the first entry.

Functions

Example

Display the graph of the function f(n) = 2n + 1 from the set of integers to the set of integers. i.e f: Z $\rightarrow$ Z.

Solution

The graph of *f* is the set of ordered pairs of the form (n, 2n + 1) where n is an integer. This graph is displayed in Figure III.

Example

Display the graph of the function f(x) = x^2 from the set of integers to the set of integers. i.e. f: Z $\rightarrow$ Z

Solution

The graph of *f* is the set of ordered pairs of the form (x, f(x)) = (x, x^2) where x is an integer. This graph is displayed in Figure III.

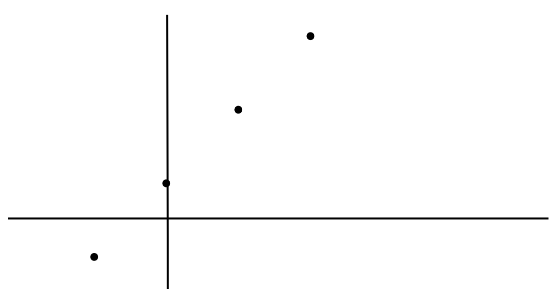


Figure II **The Graph of the Function *f(n) = 2n + 1* from Z to Z.**

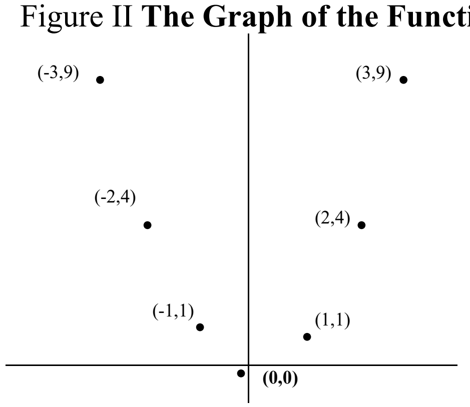


Figure III **The Graph of *f(x) = x^2 from Z to Z.***

Functions

Activity 1

1. Which of the following set of ordered pairs represent a function? Explain your answer.

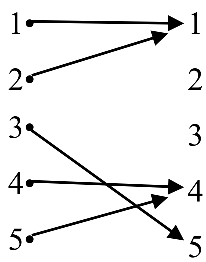
   (a) {(1, 4), (2, 3), (3, 2), (5, 3)}   (b) {(2, 5), (4, 1), (3, 3), (2, 2)}

2. Which of the following set of ordered pairs represent a function? Explain your answer.

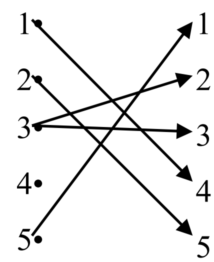
   (a) {(4, 2), (1, 1), (0, 0), (4, -2)}   (b) {(1, 3), (2, 3), (3, 3), (4, 3)}

3. Which of the following arrow diagrams represent a function?

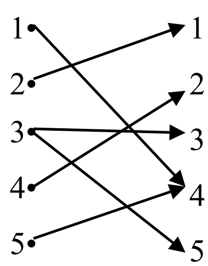
   (a)



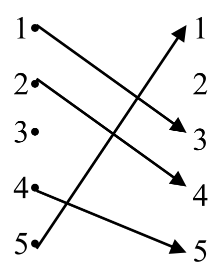
   (b)



   (c)



   (d)



4. Given the set of A = {1, 2, 3, 4, 5}; let f:A $\rightarrow$ A. For each function given, draw the set A twice, showing each element of A, then draw arrows from the elements in the first set to those in the second that represent the function *f*.

   (a) F(x) = 6 – x   (b) f(x) = $(x – 3)^2$ (c) f(x) = 2x – 3

5. Write each function in Problem 3 as a set of ordered pairs.

6. Which of the functions in Problem 3 is a one-to-one function?

7. Given each of the following rules for *f(x)*, compute *f(-1), f(2), f(3)* for each *f(x)*.

   (a) *f(x) = 3x -5*   (b) *f(x) = 10x – 2x*   (c) *f(x) =* x^2 *- x*   (d) *f(x) = 3*

8. Determine whether each of the functions in problem 7, with domain {-1, 0, 1, 2, 3}, is one-to-one.

9. Draw a graph of each of the functions in problem 7 assuming the domain is {-1, 0, 1, 2, 3} in each case.

10. Which of the following functions from Z to Z is one-to-one? onto?

    (a) *f(n)= n – 1*   (b) *f(n) =* n^2 *+ 1*   (c) *f(n)=* n^3 (d) *f(n0 = [n/2]*

Functions

Activity 2

1. Which of the following tables define y as a function of x?

(a)

x	1	2	3	2
y	4	3	2	1

(b)

x	1	-1	3	0
y	1	3	2	1

(c)

x	-2	-1	1	2
y	4	1	1	4

(d)

x	1	1	1	1
y	1	2	3	4

2. Let S = {-1, 0, 2, 4, 7}. Find $f(S)$ if
   (a) $f(x) = 1$ (b) $f(x) = 2x + 1$ (c) $f(x) = [x/5]$ (d) $f(x)= \lfloor (x^2 + 1)/3 \rfloor$

3. Let $f(x) = 2x$. What is
   (a) $f(Z)$ (b) $f(N)$ (c) $f(R)$?

4. Give an example of a function from N to N that is
   (a) one-to-one but not onto.
   (b) onto but not one-to-one.
   (c) both onto and one-to-one (but different from the identity function).
   (d) neither one-to-one or onto.

5. Find $f \circ g$ and $g \circ f$ where $f(x) = x^2 + 1$ and $g(x) = x + 2$ are functions from R to R.

6. – 10.
   Identify or create five (5) problems on functions and then solve the problems.

Theme III: Organizing and Representing Numbers and Data

Lesson 4: Graphs

Do You Know?

Graphs are discrete mathematical structures consisting of vertices (nodes or points) and edges (line segments) that connect these vertices. There are many different types of graphs based on the kind, type, and number of vertices and edges. Problems in many areas can be solved using graphs as models. This lesson will begin by introducing, defining, and giving examples of some of the most frequently used graphs.

Definition 1 A **simple undirected graph** $G = (V, E)$ consists of V, a nonempty set of **vertices**, and E, a set of unordered pairs of distinct elements of V called **edges**.

We cannot use a pair of vertices to specify an edge of a graph when multiple edges are present. This makes the formal definition of multigraphs somewhat complicated.

Definition 2 A **undirected multigraph** $G = (V, E)$, which consists of a set V of vertices, a set E of undirected edges, and a function f from E to $\{\{u, v\} \mid u, v \in V, u \neq v\}$. The edges e_1 and e_2 are called **multiple** *or* **parallel edges** if $f(e_1) = f(e_2)$.

Definition 3 A **pseudograph** or multigraph with a loop is G = (V, E), which consists of a set V of vertices, a set E of edges, and a function f from E to $\{\{u, v\} \mid u, v \in V\}$. An edge is a *loop* if $f(e) = \{u, u\} = \{u\}$ for some $u \in V$.

The reader should note that multiple edges in a pseudograph are associated to the same pair of vertices. However, we will say that $\{u, v\}$ is an edge of a graph $G = (V, E)$ if there is al least one edge e with $f(e) = \{u, v\}$. We will not distinguish between the edge e and the set $\{u, v\}$ associated to it unless the identity of individual multiple edges is important.

To summarize, pseudographs are the most general type of undirected graphs since they may contain loops and multiple edges. Multigraphs are undirected graphs that may contain multiple edges but may not have loops. Finally, simple graphs are undirected graphs with no multiple edges or loops.

Definition 4 A **directed graph** G = (V, E) consists of a set of vertices V and a set of edges E that are ordered pairs of elements V.

Graphs

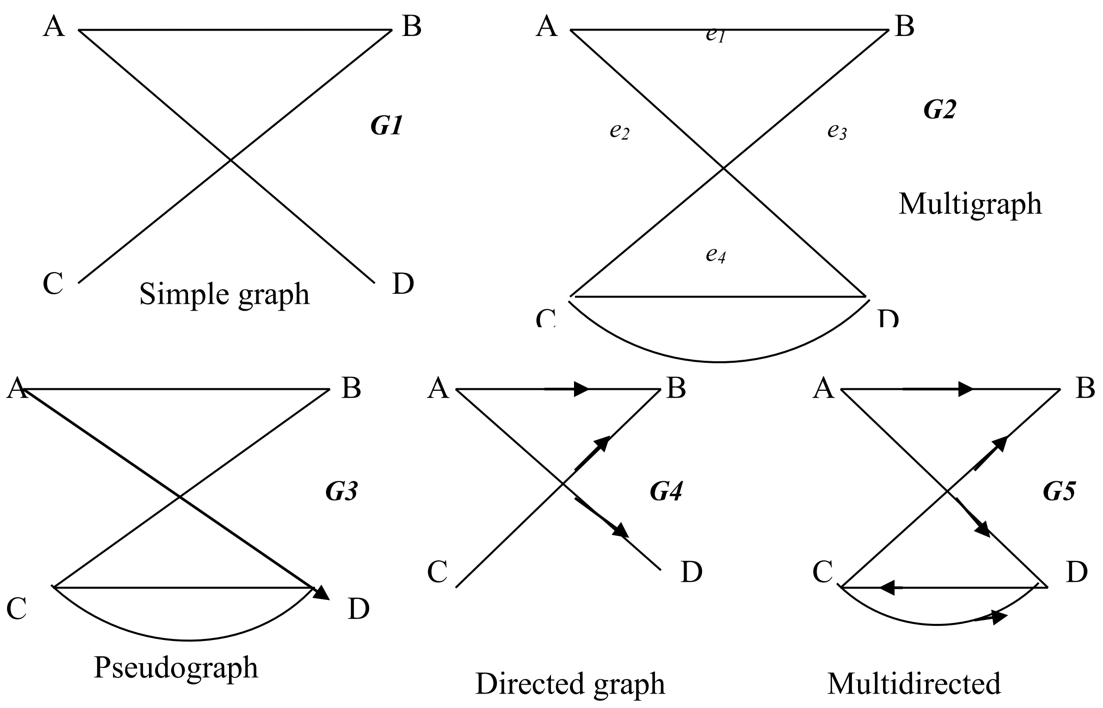
Definition 5 A **directed multigraph** G = (V, E) consists of a set V of vertices, a set E of edges, and a function *f* from E to $\{(u, v) \mid u, v \in V\}$. The edges e_1 and e_2 are **multiple edges** if $f(e_1) = f(e_2)$.

The reader should note that multiple directed edges are associated to the same pair of vertices. However, we will say that (*u, v*) is an edge of G = (V, E) as long as there is at least one edge *e* with $f(e) = (u, v)$. We will not make the distinction between the edge *e* and the ordered pair (*u, v*) associated to it unless the identity of individual multiple edges are important. This terminology for the various types of graphs makes clear whether the edges of a graph are associated to ordered or unordered pairs, whether multiple edges are allowed, and whether loops are allowed.

Table A **Types of Graphs**

Type	***Edges***	***Multiple Edges Allowed?***	***Loops Allowed?***
Simple graph	Undirected	No	No
Multigraph	Undirected	Yes	No
Pseudograph	Undirected	Yes	Yes
Directed graph	Directed	No	Yes
Directed multigraph	Directed	Yes	Yes

Figure I Some Pictures of Graphs



Graphs

In a graph G(V, E), vertices *u* and *v* are said to be **adjacent** if there is an edge $e=\{u, v\}$. In such a case, *u* and *v* are called the **endpoints** of *e*, and *e* is said to **connect** *u* and *v*. Also, the edge *e* is said to be **incident** on each of its endpoints *u* and *v*. Graphs are pictured by diagrams in the plane in a natural way. Specifically, each vertex *v* in V is represented by a dot (or small circle), and each edge $e = \{v_1, v_2\}$ is represented by a line or curve which connects its endpoints v_1 and v_2.

Subgraphs

Consider a graph G = G(V, E). A graph H = H(V′, E′) is called a **subgraph** of G if the vertices and edges of H are contained in the vertices and edges of G, that is, if $V' \subseteq V$ and $E' \subseteq E$. In particular:

• A subgraph H(V′, E′) of G(V, E) is called the subgraph **induced** by its vertices V′ if its edge set E′ contains all edges in G whose endpoints belong to vertices in H.
• If *v* is a vertex in G, the G – *v* is the subgraph of G obtained by deleting *v* from G and deleting all edges in G which contain *v*.
• If *e* is an edge in G, then G – *e* is the subgraph of G obtained by simply deleting the edge *e* from G.

Degree of a Vertex (undirected graph)

The **degree** of a vertex *v* in a graph G, written deg (*v*), is equal to the number of edges in G which contain *v*, that is, which are incident on *v*. Since each edge is counted twice in **counting** the degrees of the vertices of G, we have the following simple but important result.

Theorem A The sum of the degrees of the vertices of an undirected graph G is equal to twice the number of edges in G.

A vertex of degree zero is called an **isolated** vertex. Directed graphs have in-degrees and end-degree.

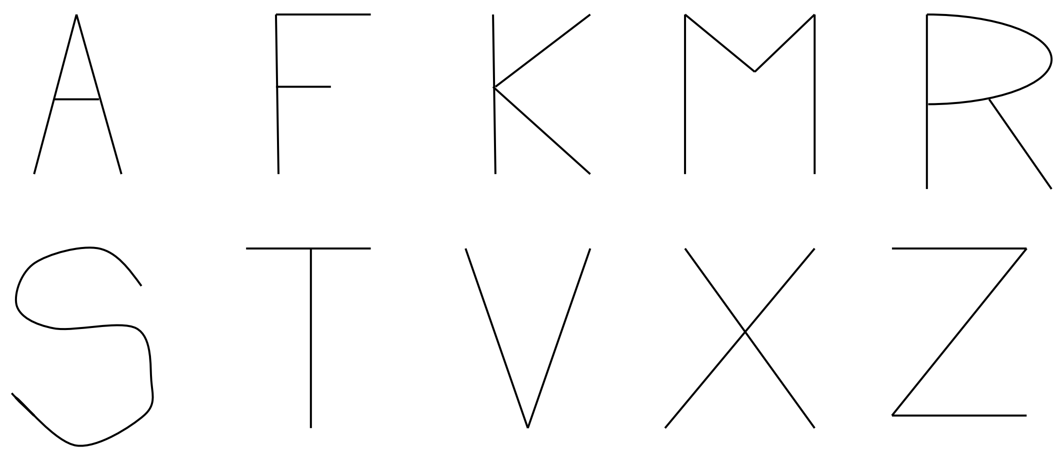
Isomorphic Graphs

Graphs G = G(V, E) and G* = H(V*, E*) are said to be **isomorphic** if there exists a one-to-one correspondence $f: V \longrightarrow V^*$ such that $\{u, v\}$ is an edge of G*. Normally, we do not distinguish between isomorphic graphs (even though their diagrams may "look different").

Graphs

Figure II gives ten graphs pictured as letters. We note that F and T are isomorphic graphs. M, S, V, and Z are also isomorphic.

Figure II



Paths; Connectivity

A **path** in a multigraph G consists of an alternating sequence of vertices and edges of the form

$$v_0, e_1, v_1, e_2, v_2, \ldots , e_{n-1}, v_{n-1}, e_n, v_n$$

where each edge e_i contains the vertices v_{i-1} and v_i (which appear on the sides of e_i in the sequence). The number n of edges is called the **length** of the path. When there is no ambiguity, we denote a path by its sequence of vertices $(v_0, v_1,\ldots,v_n)$. The path is said to be **closed** if $v_0 = v_n$. Otherwise, we say the path is from v_0 to v_n, or **between** v_0 and v_n, or **connects** v_0 to v_n.

A **simple path** is a path in which all vertices are distinct. A path in which all edges are distinct will be called a **trail.** A **cycle** is a closed path of length 3 or more in which all vertices are distinct except $v_0 = v_n$. A cycle of length k is called a ***k*-cycle**

By eliminating unnecessary edges, it is not difficult to see that any path from a vertex u to a vertex v can be replaced by a simple path from u to v. We state this result formally.

Theorem B There is a path from a vertex u to a vertex v if and only if there exists a simple path from u to v.

Graphs

Connected Graph – Connected Components – Isolated Vertex

A graph G, is **connected** if there is a path between any two of its vertices. The graph in Figure III, is connected, but the graph in Figure IV is not connected since, for example, there is no path between vertices D and E.

Figure III

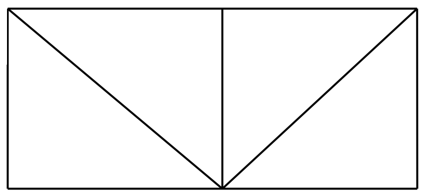
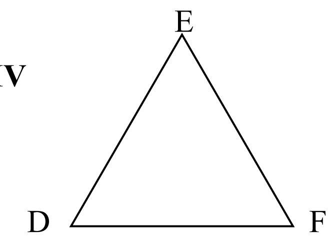


Figure IV



Suppose G is a graph. A connected subgraph H of G is called a **connected component** of G if H is not contained in any larger connected subgraph of G. It is intuitively clear that any graph G can be partitioned into its connected components. For example, the graph G in Figure IV has three (3) components DEF, JK, and L.

Labeled and Weighted Graphs

A graph G is called a **labeled graph** if its edges and/or vertices are assigned data of one kind or another. In particular, G is called a **weighted graph** if each edge *e* of G is assigned a nonnegative number *w(e)* called the **weight** or **length** of *v*. Figure V shows a weighted graph where the weight of each edge is given in the obvious way. The weight (or length) of a path in such a weighted graph G is defined to be the sum of the weights of the edges in the path. One important problem in graph theory is to find a **shortest path**, that is, a path of minimum weight (length), between any two given vertices. The length of a shortest path between P and Q is 14.

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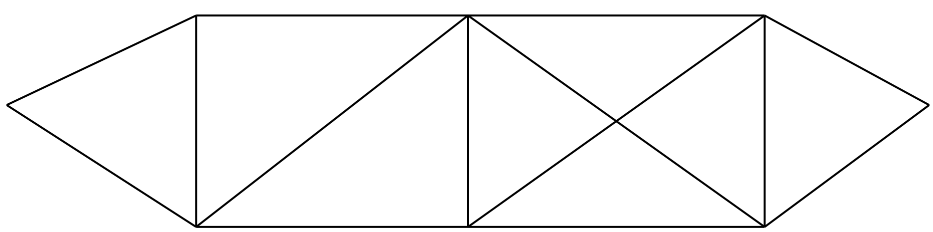


Figure V

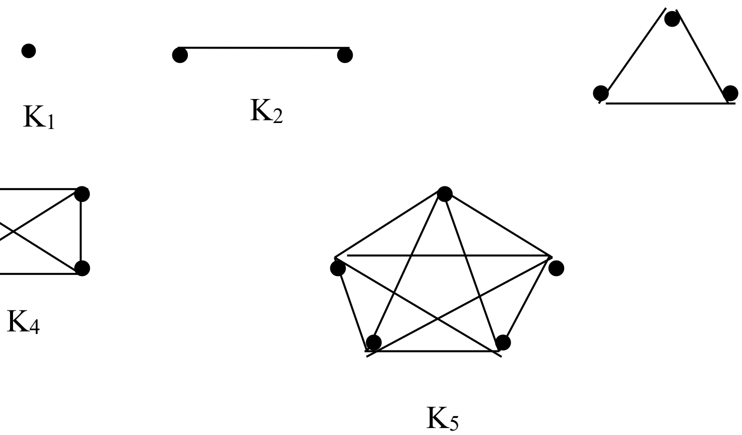
Complete, Regular, Bipartite and Tree Graphs

There are many different types of graphs. This section considers four of them: complete, regular, bipartite, and tree graphs.

Graphs

Complete Graphs
A graph g is said to be **complete** if every vertex in g is connected to every other vertex in G. Thus a complete graph G must be connected. The complete graph with *n* vertices is denoted by K_n. Figure VI shows the graphs K_1 through K_5.

Figure VI



Regular Graphs

A graph G is **regular of degree *k*** or ***k*-regular** if every vertex has degree *k*. In other words, a graph is regular if every vertex has the same degree.
The connected regular graphs of degrees 0, 1, or 2 are easily described. The connected 0-regular graph is the trivial graph with one vertex and no edges. The connected 1-regular graph is the graph with two vertices and one edge connecting them. The connected 2-regular graph with *n* vertices is the graph which consists of a single *n*-cycle. See Figure VII.

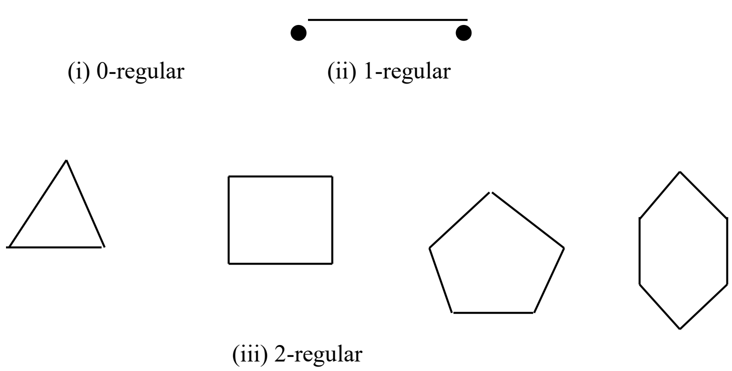
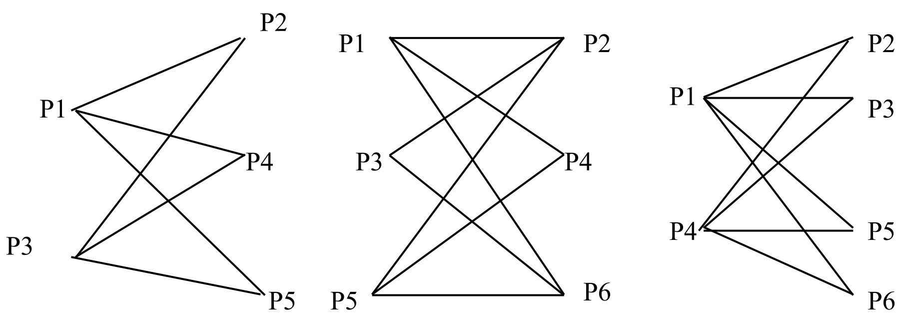


Figure W

Graphs

Bipartite Graphs

A graph G is said to be **bipartite** if its vertices V can be partitioned into two subsets M and N such that each edge of G connects a vertex of M to a vertex of N. By a **complete bipartite** graph, we mean that each vertex of M is connected to each vertex of N; this graph is denoted by $K_{m,n}$ where m is the number of vertices in N, and , for standardization, we will assume $m \leq n$. Figure VIII shows the graphs $K_{2,3}$, $K_{3,3}$, $K_{2,4}$. Clearly, the graph $K_{m,n}$. has mn edges.



$K_{2,3}$ $K_{3,3}$ $K_{2,4}$

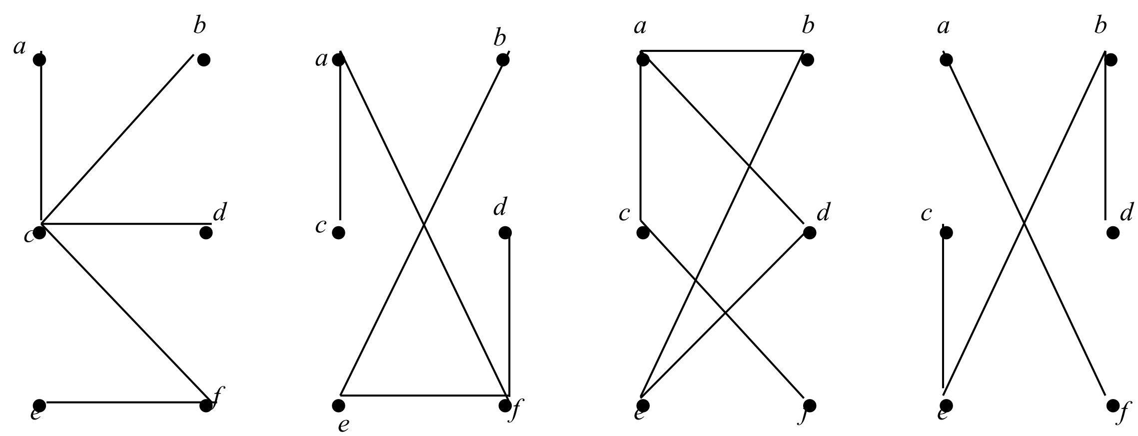
Figure V

Tree Graphs

A graph T is called a **tree** if T is connected and T has no cycles. Examples of trees are shown in Figure IX.

Any connected graph that contains no simple circuits is a tree. Trees often defined as undirected graphs with the property that there is a unique simple path between every pair of vertices.

Figure IX Examples of Trees and Graphs That Are Not Trees



G1 G2 G3 G4

Graphs

Activity 1

1. In Figure I, which graphs are undirected graphs?

2. In Figure I, which graphs are directed graphs?

3. In Figure I, which graphs have loops? For each such graph, identify the vertex where the loop is located.

4. In Figure I, Graph G3, what is the degree of Vertex A? B? C? D?

5. In Figure I, Graph G3, Name the vertices that are adjacent to the following vertex:

   (a) Vertex A: ___ ___ ___ ___ (b) Vertex B: ___ ___ ___ ___
   (c) Vertex C: ___ ___ ___ ___ (d) Vertex D: ___ ___ ___ ___

6. In Figure II, Graph G2, name the vertices that are incident to the following edge:

   (a) Edge e_1: ___ ___ ___ (b) Edge e_2: ___ ___ ___
   (c) Edge e_3: ___ ___ ___ (d) Edge e_4: ___ ___ ___
   (e) Edge e_5: ___ ___ ___

7. List two (2) reasons why in Figure II,

   (a) A and R are isomorphic. (b) R and X are isomorphic.
   (c) F and T are isomorphic.

8. In Figure III, list two (2) paths from P_1 to P_3

   (a) that are length 2. (b) that are length 3. (c) that are length 4.

9. In Figure III, list a path that is

   (a) closed and not a cycle. (b) a cycle.
   (c) a simple path. (d) a trail.

10. In Figure III, identify

   (a) two connected graphs. (b) a graph with two (2) components
   (c) an isolated vertex

Graphs

Activity 2

1. In Figure IV,
   (a) How many paths can you identify from A_4 to Q?
   (b) What is the length of each path?
   (c) What is the shortest path from A_4 to Q? What is its length? (List the vertices.)
   (d) What is the longest path from A_4 to Q? What is its length? (List the vertices.)

2. See Figure V, draw K_6.

3. See Figure VI, draw three (3) 4-regular graphs.

4. See figure VII, draw two (2) bipartite graphs,
   (a) $K_{1,5}$ (b) $K_{2,5}$

5. In Figure VIII,
   (a) Which graphs are trees?
   (b) Which graphs are not trees?
   (c) In G_1, what one edge could you remove to create two subgroups H1 and H2?
   (d) List the vertices and edges of H1, then of H2.

6. – 10.
   Identify or create five (5) problems about graphs, then solve them.

Theme III: Organizing and Representing Numbers and Data

Lesson 5: Matrices

Do You Know?

Matrices are mathematical structures that are used in many academic disciplines and they are used in many everyday applications. By definition, a **matrix** is a rectangular array of numbers or data. Example A: The matrix A = $\begin{bmatrix} 1 & 3 & 5 & -7 \\ 2 & 0 & 4 & 6 \\ 3 & 5 & 7 & 1 \end{bmatrix}$

A matrix with m rows and n columns is called an **m x n matrix**. The plural word for matrix is matrices. A matrix with the same number of rows as columns (an n x n matrix) is called a **square matrix**.

Two matrices are the **same size** if each has the same number of rows as columns.

Two matrices are **equal** if they are the same size and the corresponding entries in every position are equal.

We now introduce some more properties about matrices. Boldface uppercase letters will be used to represent matrices.

Definition 2 Let

A = $[a_{ij}]$

The *i*th row of A is the 1 x n matrix $[a_{i1}, a_{i2}, \ldots, a_{in}]$. The *j*th column of A is the n x 1 matrix

The **(i, j)th element** or entry of A is the element a_{ij}, that is, the number in the ith row and jth column of A. a convenient shorthand notation for expressing the matrix A is to write a = $[a_{ij}]$, which indicates that A is the matrix with its (i, j)th element equal a_{ij}.

Matrices

A matrix having only one row is called a row matrix (or row vector). Similarly, a matrix having only one column is called a **column matrix** (or column vector).

Since many matrices have as their entries numbers, there are well-define arithmetic operations for matrices.

4. **Addition of Matrices**. If A = (a_{jk}) and B = (b_{jk}) have the same **order** (size) we define the sum of A and B and A + B = ($a_{jk} + b_{jk}$).

Example 1. If $A = \begin{bmatrix} 2 & 1 & 4 \\ -3 & 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -5 & 1 \\ 2 & 1 & 3 \end{bmatrix}$ then

$$A + B = \begin{bmatrix} 2 + 3 & 1 - 5 & 4 + 1 \\ -3 + 2 & 0 + 1 & 2 + 3 \end{bmatrix} = \begin{bmatrix} 5 & -4 & 5 \\ -1 & 1 & 5 \end{bmatrix}$$

Note that the commutative and associative laws for addition are satisfied by matrices, i.e. for any matrices A, B, C of the same size (or order)

$$A + B = B + A, \quad A + (B + C) = (A + B) + C$$

2. **Subtraction of Matrices**. If A = (a_{jk}), B = (b_{jk}) have the same size (order), we define the **differences** of A and B as A – B = ($a_{jk} - b_{jk}$).

Example 2. If A and B are the matrices of Example 1, then

$$A - B = \begin{bmatrix} 2 - 3 & 1 + 5 & 4 - 1 \\ -3 - 2 & 0 - 1 & 2 - 3 \end{bmatrix} = \begin{bmatrix} -1 & 6 & 3 \\ -5 & -1 & -1 \end{bmatrix}$$

5. **Multiplication of a Matrix by a Number**. If A = (a_{jk}) and λ is any number [or scalar], we define the product of A by c as cA = Ac = (ca_{jk}).

Example 2. If A is the matrix of Example 1 and λ = 4, then

$$cA = 4\begin{bmatrix} 2 & 1 & 4 \\ -3 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 4 & 16 \\ -12 & 0 & 8 \end{bmatrix}$$

Matrices

4. **Multiplication of Matrices.** If A = (a_{jk}) is an $m \times n$ matrix while B = (b_{jk}) is an $n \times p$ matrix, then we define the **product** A $\cdot$ B or AB of A and B as the matrix C = (c_{jk}) where

$$C_{jk} = \sum_{l=1}^{n} a_{jl}b_{lk}$$

and where C is of order $m \times p$.

Note that matrix multiplication is defined if and only if the number of columns of A is the same as the number of rows of B. Such matrices are sometimes called **compatible (**for multiplication**).**

Example 4

$$\text{Let A} = \begin{pmatrix} 2 & 1 & 4 \\ -3 & 0 & 2 \end{pmatrix}, \text{D} = \begin{pmatrix} 3 & 5 \\ 2 & -1 \\ 4 & 2 \end{pmatrix}. \quad \text{Then}$$

$$\text{AD}= \begin{pmatrix} (2)(3)+(1)(2)+(4)(4) & (2)(5)+(1)(-1)+(4)(2) \\ (-3)(3)+(0)(2)+(2)(4) & (-3)(5)+(0)(-1)+(2)(2) \end{pmatrix} = \begin{pmatrix} 24 & 17 \\ -1 & -11 \end{pmatrix}$$

Note that the general AB ≠ BA. i.e. the commutative law for multiplication of matrices is not satisfied in general. However, the associative and distributive laws are satisfied, i.e.

A(BC) = (AB)C, A(B + C) = AB + AC, (B + C)A = BA + CA

A matrix A can be multiplied by itself if and only if it is a square matrix.
The product A $\cdot$ A can in such case be written A^2. Similarly we define powers of a square matrix, i.e. $A^3 = A \cdot A^2$, $A^4 = A \cdot A^3$, etc.

5. **Transpose of a Matrix.** If we interchange rows and columns of a matrix A, the resulting matrix is called the **transpose** of A and is denoted by A^T.
In symbols, if A = (a_{jk}) then A^T = (a_{kj}).

Example 5

$$\text{The transpose of A} = \begin{pmatrix} 2 & 1 & 4 \\ -3 & 0 & 2 \end{pmatrix} \text{ is}$$

$$A^T = \begin{pmatrix} 2 & -3 \\ 1 & 0 \\ 4 & 2 \end{pmatrix}$$

We can prove that

$$(A + B)^T = A^T + B^T, (AB)^T = B^TA^T, (A^T)^T = A$$

Matrices

6. **Symmetric and Skew-Symmetric Matrices.** A square matrix A is called **symmetric** if $A^T = A$ and **skew-symmetric** if $A^T = -A$.

Example 6

The matrix $E = \begin{pmatrix} 2 & -4 \\ -4 & 3 \end{pmatrix}$ is symmetric while $F = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$ is skew-symmetric.

7. **Unit Matrix.** A square matrix in which all elements of the principal diagonal are equal to 1 while all other elements are zero is called the **unit matrix** and is denoted by I. An important property of I is that

$$AI = IA = A,\ I^n = I,\ n = 1, 2, 3, \ldots$$

The unit matrix plays a role in matrix algebra similar to that played by the number one in ordinary algebra.

8. **Zero or Null Matrix.** A matrix whose elements are all equal to zero is called the **null** or **zero matrix** and is often denoted by O or simply 0. For any matrix A having the same order as 0 we have

$$A + 0 = 0 + A = A$$

Also if A and 0 are square matrices, then

$$A0 = 0A = 0$$

The zero matrix plays a role in matrix algebra similar to that played by the number zero of ordinary algebra.

9. **Principal Diagonal and Trace of a Matrix.** If $A = (a_{jk})$ is a square matrix, then the diagonal which contains all elements a_{jk} for which j = k is called the **principal** or **main** diagonal and the sum of all such elements is called the **trace** of A.

Example 7

The principal or main diagonal of the matrix

$$\begin{pmatrix} 5 & 2 & 0 \\ 3 & 1 & -2 \\ -1 & 4 & 2 \end{pmatrix}$$

is {5, 1, 2}, and the trace of the matrix is 5 + 1 + 2 = 8.

A matrix for which $a_{jk} = 0$ when $j \neq k$ is called a **diagonal matrix.**

Matrices

Matrices and Graphs

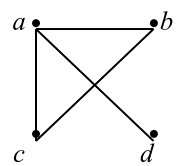
There are two types of matrices commonly used to represent graphs. One type is an adjacency matrix. The other type is an incidence matrix. We will only discuss the adjacency matrix. An **adjacency matrix** is defined as a matrix whose rows and columns represent the vertices of a matrix and that the a_{ij} entry is a 1 or 0, depending upon whether or not the i^{th} vertex is adjacent to the j^{th} vertex.

Example 8

Use an adjacency matrix to represent the graph shown in Figure A.

Figure A

A Simple Graph



$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Solution

We order the vertices as *a, b, c, d*. The matrix representing this graph is

It is interesting to observe that the adjacency matrix of a graph is unique; it only defines one graph or a graph isomorphic to it. Thus, given an adjacency matrix, one can determine the graph (or isomorphic graph) that it represents.

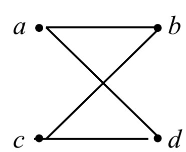
Example 9

Draw a graph with the adjacency matrix

$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

with respect to the ordering vertices *a, b, c, d*.

Figure B
Graph with the Given Adjacency Matrix



Adjacency matrices can also be used to represent undirected graphs with loops and with multiple edges. A loop at the vertex a_i is represented by a 1 at the (i, i)th position of the adjacency matrix. When multiple edges are present, the adjacency matrix is no longer a zero-one matrix, since the (i, j)th entry of this matrix equals the number of edges that are associated to $\{a_i, a_j\}$. All undirected graphs, including multigraphs and pseudographs, have symmetric adjacency matrices.

Matrices

Activity 1

Given A = $\begin{pmatrix} 2 & -1 \\ 4 & 3 \end{pmatrix}$**, B =** $\begin{pmatrix} -1 & 1 \\ 2 & -4 \end{pmatrix}$**, C =** $\begin{pmatrix} 1 & 4 \\ -2 & -1 \end{pmatrix}$

1. Find (a) A + B (b) A – B

2. Find (a) 2A – 3C (b) 3A + 2B – 4C

3. Find (a) AB = (b) BA =
   Does AB = BA?

4. Find (a) (AB)C = (b) A(BC) =
   Is (AB)C = A(BC)?

5. Given A = $\begin{pmatrix} 1 & 0 & 1 & 0 \\ 2 & 3 & 4 & 5 \\ 4 & 4 & 4 & 4 \end{pmatrix}$ What is A^T?

6. What is $A \cdot A^T$?

7. Write as a matrix
   (a) A, the 5 x 5 zero matrix.
   (b) B, the 5 x 5 identity matrix.

- **A =** $\begin{bmatrix} 1 & -2 & 3 \\ 0 & 4 & 5 \end{bmatrix}$**, B =** $\begin{bmatrix} 4 & 6 & 8 \\ 1 & -3 & -7 \end{bmatrix}$**, C =** $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 0 & 6 & 0 \\ 7 & 0 & 8 & 0 \end{bmatrix}$

8. Use A, B, and C from (•). What is $(A^T)^T$, $(B^T)^T$, and $(C^T)^T$? What general conclusion do you think might be true?

9. Use A and B from (•). Calculate $(A + B)^T$ and $(A^T + B^T)$; are they the same?

10. Use A and C from (•). Calculate $(AC)^T$ and $A^T C^T$; are they the same?

Matrices

Activity 2

Given A = $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix}$ **C=** $\begin{bmatrix} 0 & -2 & 1 \\ 2 & 3 & -2 \\ -1 & -1 & 1 \end{bmatrix}$

B = $\begin{bmatrix} 2 & -3 & 5 \\ -3 & 6 & 7 \\ 5 & 7 & 8 \end{bmatrix}$ **D =** $\begin{bmatrix} 0 & 3 & -4 \\ -3 & 0 & 5 \\ 4 & -5 & 0 \end{bmatrix}$

1. What are the elements of the principal diagonal of
   (a) Matrix A? (b) Matrix B? (c) Matrix C? (d) Matrix D?

2. What are the elements of the other diagonal of
   (a) Matrix A? (b) Matrix B? (c) Matrix C? (d) Matrix D?

3. What is the trace of
   (a) Matrix A? (b) Matrix B? (c) Matrix C? (d) Matrix D?

4. Which matrix, if any, is symmetric? Skew-symmetric?

5. Multiply A · C and get the matrix D (D = AC). Based on D, what would you say about A and C?

6. – 10.
   Identify, create, or construct five (5) problems involving matrices and then solve them.

Theme: IV

Theme IV: Logic and Mathematical Reasoning

Lesson 1: Propositions

Do You Know?

The rules of logic give precise meaning to mathematical statements. These rules are used to distinguish between valid and invalid mathematical reasoning. Since a major goal of this theme is to teach the reader how to understand and how to do correct mathematical reasoning, we begin our study of this theme with an introduction to logic. In addition to its importance in understanding mathematical reasoning, logic had numerous applications in many other ways. These rules are used in the design of computer circuits, the construction of computer programs, the verification of the correctness of arguments in legal areas, and in many other ways.

Propositions

We begin this Theme with an introduction to the basic building blocks of logic-propositions. A **proposition** is a statement that is either true or false, but not both.

Example 1

All the following statements are propositions.

1. Washington, D.C., is the capitol of the United States of America.
2. Paris is the capital of France.
3. $1 + 1 = 2$.
4. $2 + 2 = 3$.

Propositions 1, 2, and 3 are true, whereas 4 is false.

Some sentences that are not propositions are given in the next example.

Example 2

Consider the following sentences.

1. What time is it?
2. Read this carefully.
3. $x + 1 = 2$.
4. $x + y = z$.

Sentences 1 and 2 are not propositions because they are not statements. Sentences 3 and 4 are not propositions because they are neither true nor false, since the variables in these open sentences have not been assigned values.

Propositions

Lowercase letters are used to denote propositions. The conventional letters used for propositions are *p, q, r, s,* .... The **truth value** of a proposition is true, denoted by T, if it is a true proposition and false, denoted by F, if it is a false proposition.

We now turn our attention to methods for producing new propositions from those that we already have. Many mathematical statements are constructed by combining one or more propositions. New propositions are called **compound propositions**. They are formed from existing propositions using logical operators.

Definition 1 Let p be a proposition. The statement

"It is not the case that p."

is another proposition, called the **negation** of p. The negation of p is denoted by $\neg p$.
The proposition $\neg p$ is read "not p."

Find the negation of the proposition: "Today is Friday." "Today is not Friday."

Definition 2 Let p and q be propositions. The proposition *"p and q,"* denoted by $p \wedge q$, is the proposition that is true when both p and q are true and false otherwise. The proposition $p \wedge q$ is called the **conjunction** of p and q.

Example 3 Find the conjunction of the propositions p and q where p is the proposition "Today is Friday." And q is the proposition "It is raining today."

Solution The conjunction of these propositions, $p \wedge q$, is the proposition "Today is Friday and it is raining today." This proposition is true on rainy Fridays and is false on any day that is not a Friday and on Fridays when it does not rain.

Definition 3 Let p and q be propositions. The proposition "p or q," denoted by $p \vee q$, is the proposition that is false when p and q are both false and true otherwise. The proposition $p \vee q$ is called the **disjunction** of p and q.

The use of the connective **or** in a disjunction corresponds to one of the two ways the word **or** is typically used, namely in an inclusive way. A disjunction is true when either of the two propositions in it is true or when both are true. For instance, the inclusive or is being used in this statement:

"Students who have taken calculus or physics can take this class."

What is meant is that students who have taken both calculus and physics can take the class, as well as students who have taken just one of the two subjects.

Propositions

Logic

Similarly, when a menu at a restaurant states, “Soups or salad comes with an entrée,” the restaurant almost always means that customers can have either soup or salad, but not both. Hence, this is an exclusive, rather than an inclusive, or.

Example 5 What is the disjunction of p and q, $p \vee q$, is the proposition

“Today is Friday or it is raining today.”

This proposition is true on any day that is either a Friday or a rainy day (including rainy Fridays). It is only false on days that are not Fridays when it also does not rain.

As we previously remarked, the use of the connective or in a disjunction corresponds to one of the two ways the word **or** is typically used, namely in an inclusive way. Thus, a disjunction is true when either of the two propositions in it is true or when both are true. Sometimes, we use or in an exclusive sense. When the exclusive *or* is used to connect the propositions p and q, the proposition “p or q (but not both)” is obtained. This proposition is true when p is true and q is false, or vice versa, and it is false when both p and q are false and when both are true.

Definition 4 Let p and q be propositions. The **exclusive or** of p and q, denoted by $p \oplus q$, is the proposition that is true when exactly one of p and q is true and is false otherwise.

Definition 5 Let p and q be propositions. The **implication** $\boldsymbol{p \rightarrow q}$ is the proposition that is false when p is true and q is false and true otherwise. In this implication p is called the **hypothesis** (or premise) and q is called the **conclusion** (or consequence).

Because implications arise in many places in mathematical reasoning, a wide variety of terminology is used to express $p \rightarrow q$. Some of the more common ways of expressing this implication are:

- “if p, then q”
- “p implies q”
- “if p, q”
- “p only if q”
- “p is sufficient for q”
- “q if p”
- “q whenever p”
- “q is necessary for p”

Note that $p \rightarrow q$ is false only in the case that p is true but q is false, so that it is true when both p and q are true, and when p is false (no matter what truth value q has).

Propositions

The way we have defined implications is more general than the meaning attached to implications in the ordinary sense. For instance, the implication

"If it is sunny today, then we will go to the beach."

is an implication used in normal language, since there is a relationship between the hypothesis and the conclusion. Further, this implication is considered valid unless it is indeed sunny today, but we do not go to the beach.

Logic

There are some related implications that can be formed from $p \to q$. the proposition
q $\to p$ is called the **converse** of $p \to q$.
$\sim p \to \sim q$ is called the **inverse** of $p \to q$. The **contrapositive** of $p \to q$ is the proposition
$\sim q \to \sim p$.

Example 7 Find the converse, inverse, and the contrapositive of the implication

"If today is Thursday, then I have a test today."

Solution

The converse is, "If I have a test today, then today is Thursday.'
The inverse is, "If today is not Thursday, then I do not have a test today."
The contrapositive of this implication is, "If I do not have a test today, then today is not Thursday."

We now introduce another way to combine propositions.

Definition 6 Let p and q be propositions. The **biconditional** $p \leftrightarrow q$ is the proposition that is true when p and q have the same truth values and is false otherwise.

Observe that the biconditional $p \leftrightarrow q$ is true precisely when both the implications $p \to q$ and q $\to p$ are true. Because of this, the terminology

"p if and only if q"

is used for this biconditional. Other common ways of expressing the proposition $p \leftrightarrow q$ are: "p is necessary and sufficient for q" and "if p then q, and conversely."

.

Propositions

Logic and Bit Operations

Computers represent information using bits. A **bit** has two possible values, namely, 0 (zero) and 1 (one). Bit comes from **b**inary dig**it**, since zeros and ones are the digits used in binary representations of numbers. A bit can be used to represent a truth value, since there are two truth values, namely *true* and *false*. As is customarily done, we will use a 1 bit to represent true and a 0 bit to represent false. That is, 1 represents T (true), 0 represents F (false).

Computer **bit operations** correspond to the logical connectives. By replacing true by a one and false by a zero in the truth tables for the operators. We will also use the notation *OR, AND,* and *XOR* for the operators $\wedge, \vee, and \oplus$.

Information is often represented using **bit strings**, which are sequences of zeros and ones. When this is done, operations on the bit strings can be used to manipulate this information.

Definition 7 A **bit string** is a sequence of zero or more bits. The **length** of this string is the number of bits in the string.

Listed below is how the bit operations are defined.

01 1011 0110	
<u>11 0001 1101</u>	
11 1011 1111	*bitwise OR*

01 1011 0110	
<u>11 0001 1101</u>	
01 0001 0100	*bitwise AND*

01 1011 0110	
<u>11 0001 1101</u>	
10 1010 1011	*bitwise XOR*

Propositions

Activity 1

1. Which of the following sentences are propositions? What are the truth values of those that are propositions?
   (a) Dakar is the capital of Senegal. (b) $2 + 3 = 5$. (c) $5 + 7 = 10$.

2. What is the negation of each of the following propositions?
   (a) Today is Thursday.
   (b) $2 + 1 = 3$.
   (c) During the rainy season it is hot and sunny.

3. Let p and q be the propositions
   p: I bought a car this week.
   q: I went to the Mosque on Friday.

   Express each of the following propositions as a regular sentence.
   (a) $\sim p$ (b) $p \vee q$ (c) $p \rightarrow q$ (d) $p \wedge q$

4. Let p and q be the propositions
   p: It is above 30˚C.
   q: It is raining.

   Write the following propositions using p and q and logical connectives.
   (a) It is above 30˚C and raining.
   (b) It is above 30˚C but not raining.
   (c) It is not above 30˚C and it is not raining.

5. Let p, q and r be the propositions
   p: You get an A on the final exam.
   q: You do every activity in this book.
   r: You get an A in this class.

   Write the following propositions using p, q and r and logical connectives.
   (a) You get an A on the final, you do every activity in this book, and you get an A in this class.
   (b) Getting an A on the final and doing every activity in this book is sufficient for getting an A in this class.
   (c) You will get an A in this class if and only if you either do every activity in this book or you get an A on the final.

6. Determine whether each of the following implications is true or false.
   (a) If $1 + 1 = 2$, then $2 + 2 = 5$.
   (b) If $1 + 1 = 3$, then $2 + 2 = 4$.
   (c) If $1 + 1 = 3$, then $2 + 2 + 5$.
   (d) If $2 + 2 = 4$, then $1 + 2 = 3$.

Propositions

Activity 1
(Continued)

7. For each of the following sentences, determine whether an inclusive *or* or an exclusive *or* is intended.
   (a) Being age 18 or a parent is required.
   (b) Lunch includes soup or salad.
   (c) To enter the country you need a passport or a national card.

8. Write each of the following statements in the form “if p then q” in sentence form. (*Hint:* Refer to the list of ways to express implications listed in this section).

   (a) I will remember to send you the address only if you send me an email message.
   (b) To be a citizen of this country, it is sufficient that you were born in the United States.
   (c) If you keep your textbook, it will be a useful reference in your future courses.

9. Write each of the following propositions in the form “p if and only if q” in sentence form.
   (a) If it is hot outside you buy an ice cream cone, and if you buy an ice cream cone it is hot outside.
   (b) For you to win the contest it is necessary and sufficient that you have the only winning ticket.
   (c) You get promoted only if you have connections, and you have connections only if you get promoted.

10. Write each of the following propositions in the form “p if and only if q” as sentences.
   (a) For you to get an A in this course, it is necessary and sufficient that you learn how to solve all activity problems.
   (b) If you read the newspaper every day, you will be informed, and conversely.
   (c) It rains if it is a weekend day, and it is a weekend day if it rains.

Propositions

Activity 2

1. State the converse, inverse, and contrapositive of each of the following implications.
   (a) If it rains today, I will swim tomorrow.
   (b) I come to class whenever there is going to be an exam.
   (c) A positive integer is a prime only if it has not divisors other than itself.

2. State the converse, inverse, and contrapositive of each of the following implications.
   (a) If it snows tonight, then I will stay at home.
   (b) I go to the beach whenever it is a sunny summer day.
   (c) When I stay up late, it is necessary that I sleep until noon.

3. Each inhabitant of a remote village always tells the truth or always lies. A villager will only give a "Yes" or a "No" response to a question a tourist asks. Suppose you are a tourist visiting this area and come to a fork in the road. One branch leads to the ruins you want to visit; the other branch leads deep into the jungle. A villager is standing at the fork in the road. What one question can you ask the villager to determine which branch to take?

4. Find the *bitwise OR, bitwise AND,* and *bitwise XOR* of each of the following pairs of bit strings.

   (a) 101 1110, 010 0001
   (b) 1111 0000, 1010 1010

5. Evaluate each of the following expressions.
   (a) 1 1000 $\wedge$ (0 1011 $\vee$ 1 1011)
   (b) (0 1111 $\wedge$ 1 0101) $\vee$ 0 1000)

Theme IV: Logic and Mathematical Reasoning

Lesson 2: Truth Tables

Do You Know?

As we stated in Lesson 1, a **proposition** is a statement that is either true or false, but not both. The truth value of statements consisting of several propositions, can be determined very easily by truth tables. In this lesson we introduce the readers to truth tables. We begin with the basic truth tables and continue to study those that are more complex.

A **truth table** displays the relationships between the truth values of propositions. Truth tables are especially valuable in the determination of the truth values of propositions constructed from simpler propositions. Table 1 displays all possible truth values of a proposition and the corresponding truth values of its negation.

The negation of a proposition can also be considered the result of the operation of the **negation operator** on a proposition. The negation operator constructs a new proposition from a single existing proposition.

TABLE 1 The Truth Table for the Negation of a Proposition

p	$\sim p$
T	**T**
F	**F**

This is the simplest of the basic truth tables. It illustrates what each of the truth tables will actually display: all of the possible true (T), false (F) possibilities with the propositions under consideration and the logical connective(s) under consideration.
We now present six (6) additional basic truth tables.

Table 2 is the truth table for the conjunction of two propositions.
Table 3 is the truth table for the disjunction of two propositions.

TABLE 2 The **Truth Table for the Conjunction of Two Propositions.**

p q	$p \wedge q$
T T	T
T F	F
F T	F
F F	F

TABLE 3 The **Truth Table for the Disjunction of Two Propositions.**

p q	$p \vee q$
T T	T
T F	T
F T	T
F F	F

Truth Tables

Table 4 is the truth table for the exclusive or of two propositions.
Table 5 is the truth table for the implication of $p \rightarrow q$.

TABLE 4 The **Truth Table for the Exclusive or of Two Propositions.**

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

TABLE 5 **The Truth Table for the Implication of $p \rightarrow q$.**

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Table 6 is the truth table for the biconditional $p \leftrightarrow q$.
Table 7 is the table for the bit operators OR, AND, and XOR.

TABLE 6 The **Truth Table for Biconditional $p \leftrightarrow q$.**

p	q	$p \leftrightarrow q$
T	T	F
T	F	F
F	T	F
F	F	T

TABLE 7 The **Truth Table for the Bit Operators OR, AND, and XOR.**

x	y	$x \vee y$	$x \wedge y$	$x \oplus y$
0	0	0	0	0
0	1	1	0	1
1	0	1	0	1
1	1	1	1	0

One should note that the first seven truth tables have been provided for two propositions only. How do the truth tables change if one considers three (3) or more simple propositions?

First observe carefully the truth table involving three (3) propositions.

TABLE 8 Truth Values for $p \wedge (q \vee r)$, Involving Three (3) Propositions.

p	q	r	$q \vee r$	$p \wedge (q \vee r)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	T	F
F	T	F	T	F
F	F	T	T	F
F	F	F	F	F

Truth Tables

Observe that the tables involving two simple propositions had four rows of truth values and Table 8 which involves three simple propositions has eight rows of truth values. How many rows of truth values do you believe that a table involving four different simple propositions would contain? (**Hint**: 2 propositions, $2^2 = 4$ rows; 3 propositions, $3^2 = 8$ rows; *n* propositions, 2^n rows.) Truth tables are valuable for helping one to learn the truth value of compound statements. Additionally, truth tables are very valuable for helping one to determine whether or not two compound statements are **logically equivalent** (have identical truth values). Truth tables are also very valuable for helping one to learn whether or not to compound proposition is a **contradiction** (all truth values are F) of a **tautology** (all truth values are T). We now present examples of such propositions.

Table 9 illustrates that $\sim p \vee q$ and $p \rightarrow q$ are logically equivalent.
Table 10 illustrates that $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ are logically equivalent.

TABLE 9 $\sim p \vee q$ **and** $p \rightarrow q$ **are logically equivalent.**

p q	~p	~p ∨ q	p → q
T T	F	T	T
T F	F	F	F
F T	T	T	T
F F	T	T	T

TABLE 10 $p \vee (q \wedge r)$ **and** $(p \vee q) \wedge (p \vee r)$ **are logically equivalent.**

p q r	q ∧ r	p ∨ (q ∧ r)	p ∨ q	p ∨ r	(p ∨ q) ∧ (p ∨ r)
T T T	T	T	T	T	T
T T F	F	T	T	T	T
T F T	F	T	T	T	T
T F F	F	T	T	T	T
F T T	T	T	T	T	T
F T F	F	F	T	F	F
F F T	F	F	F	T	F
F F F	F	F	F	F	F

Table 11 illustrates both a contradiction and a tautology.

TABLE 11 **Both a contradiction and a tautology.**

p	*~p*	*p ∨ ~p*	*p ∧ ~p*
T	**F**	**T**	**F**
F	**T**	**T**	**F**

Truth Tables

Activity 1

1. Construct a truth table for

   (a) $p \wedge \sim p$ (b) $p \vee \sim p$

2. Construct a truth table for

   (a) $(p \vee \sim q) \rightarrow q$ (b) $(p \vee \sim q) \rightarrow (p \wedge q)$

3. Construct a truth table for

   (a) $(p \rightarrow q) \rightarrow (q \rightarrow p)$ (b) $(p \rightarrow q) \leftrightarrow (\sim q \rightarrow \sim p)$

4. Construct a truth table for

   (a) $p \oplus \sim q$ (b) $\sim p \oplus \sim q$

5. Construct a truth table for

   (a) $(p \oplus q) \vee (p \oplus \sim q)$ (b) $(p + q) \wedge (p \oplus \sim q)$

6. Construct a truth table for

   (a) $(p \rightarrow q) \vee (\sim p \rightarrow q)$ (b) $(p \rightarrow q) \wedge (\sim p \rightarrow q)$

7. Construct a truth table for

   (a) $\sim p \wedge \sim q$ (b) $\sim[p \vee (\sim p \wedge q)$

   What comments would you make about (a) and (b)?

8. Construct a truth table for

   $[(\sim p \vee q) \wedge p] \rightarrow q$

9. Construct a truth table for

   $(p \rightarrow q) \wedge (p \wedge \sim q)$

10. Compare and contrast the truth tables in Problems 8 and 9.

    What are the special names of each? In Problems 1 – 9, do you observe any compound propositions that were logically equivalent, a tautology or a contradiction; if yes, identify.

Truth Tables

Activity 2

1. Construct a truth table for

   (a) $(p \vee q) \vee r$ (b) $(p \wedge q) \vee r$

2. Construct a truth table for

   (a) $(p \wedge q) \wedge r$ (b) $(p \wedge q) \wedge r$

3. Construct a truth table for

   $[\sim p \wedge (p \vee q)] \rightarrow q$
   What is the special name for this compound statement?
4. Evaluate the following expression

   $(01111 \wedge 10101) \vee 010000$
5. Evaluate

   $(01010 \oplus 11011) \oplus 01000$

Theme IV: Logic and Mathematical Reasoning

Lesson 3: Open statements, Quantifiers and Quantified Statements

Do You Know?

Statements involving variables such as “$x > 5$, $x=y + 7$ and $x+ y = n$ are frequently found In mathematical discussions and reasoning. These statements are neither true nor false when the values of the variables are not specified. They are **conditional statements** that only become propositions when the values of the variables are specified. Often we write these conditional statements with say a variable x, as p(x).

Example $p(x)$: $x> 5$

Quantified statements are statements involving variables where we know the values of the variables and can evaluate whether the statements are true or false.

Example
$x(x)$: $x> 2$ is odd, where x is a prime.

Once conditional statements with variables become quantified statements where the value(s) of the variable is known, they become propositions and can be evaluated as true or false. When writing open statements, we often use quantifiers. In this lesson we will discuss two types of quantifiers: universal and existential.

Universal Quantifiers

The words all, every, and each are called **universal quantifiers.** When these words are added to open sentences, they change them to statements that are true or false. Open statements involving quantifiers are called quantified statements.

Examples of Quantified statements:

All men have hair on their heads.
Every truck uses diesel fuel.
For each real number x, $x+ 7 = 7 +x$.

Open statements, Quantifiers and Quantified Statements

Statements written with universal quantifiers have the property that they can usually be written as an implication (conditional statement).

If a person is a man, then he has hair on his head.
If a vehicle is a truck, then it uses diesel fuel.
If a number is real, then x + 7 = 7 + x.

Existential Quantifiers

There are other quantified statements that are intended to indicate the existence of at least one case in which the statement is true. Such statements generally involve one of the following existential qualifiers: some, there exist or exists at least one.

Example S

Some men have no hair on their heads.
There exist students who study very hard.
There exists at least one student who does not attend school on a regular basis.

Negations of Quantified Statements

To be the **negation** of a quantified statement, the newly formulated statement must have truth values that are the opposite of the truth values of the original statement in every possible situation.

Statement	**Negation**
All *p* are *q*.	Some *p* are not *q*.
Some *p* are *q*.	No *p* is *q*. All *p* are not *q*.
Some *p* are not *q*.	All *p* are *q*.
No *p* is *q*.	Some *p* are *q*.

Example

Write the negation of each statement.

(a) Some girls have red hair. (At least one girl has red hair.)
(b) All apples are yellow.
(c) No physician is bald-headed.
(d) Some fishermen do not work hard. (At least one fisherman does not work hard.)

Solution:

(a) "No girl has red hair" or "All girls have hair that is not red."
(b) Some apples are not yellow.
(c) Some physicians are bald-headed. (At least one physician is bald-headed.)
(d) All fisherman work hard.

Open statements, Quantifiers and Quantified Statements

Counter Example

To show that a universally quantified statement is false, one needs to find only one case for which the statement is false; that is, one need to identify only one **counter example.** However, in showing that an existentially quantified statement is false, you must show that it is false for all possibilities. Similarly, an existentially quantified statement is true if you can find one case for which it is true, but a universally quantified statement is true only if it is true for all cases.

Let *P(x)* represent an open sentence. Then, symbolically, the **universal quantifier** is represented by $\forall$ ***x*** and the **existential quantifier** is represented by $\exists$ ***x***.

Example	Let P(x): "*x >3.*"	open sentence
	$\forall x$ (for every *x*)	universal quantifier
	$\exists x$ (there exist *x*)	existential quantifier
	$\forall x\, P(x)$	quantified universal statement
	$\exists x\, P(x)$	quantified existential statement

Using the symbols and notations, we can summarize what we have said about quantified statements and the negation of quantified statements.

TABLE 1 **Quantifiers**

Quantified Statement	When True?	When False?
$\forall x\, P(x)$	*P(x)* is true for every *x*.	There is an *x* for which *P(x)* is false.
$\exists x\, P(x)$	There is an *x* for which *P(x)* is true.	*P(x)* is for every *x*.

TABLE 1 **Negating Quantifiers**

Quantified Negation	Equivalent Statement	When is a Negation True?	When False?
$\sim\exists x\, P(x)$	$\forall x \sim P(x)$	*P(x)* is false for every *x*.	There is an *x* for which *P(x)* is true.
$\sim\forall x\, P(x)$	$\exists x \sim P(x)$	There is an *x* for which *P(x)* is false.	*P(x)* is true for every *x*.

Open statements, Quantifiers and Quantified Statements

Activity 1

1. Let P(x) be the open sentence "$x \leq 7$"
   What is the truth value of the following?
   (a) P(-7) (b) P(0) (c) P(7) (d) P(14)

*** P(x): "x spends more than 7 hours in class every week (where x is a student)."**

2. Using P(x) in (*), express as a written quantified statement
   (a) $\exists x\ P(x)$ (b) $\forall x\ P(x)$

3. Using P(x) in *, express as a written quantified statement.
   (a) $\exists x \sim P(x)$ (b) $\forall x \sim P(x)$

4. Rewrite the following quantified statement using the symbols $\exists$ and $\forall$.
   (a) For all x, $x^2 = 64$. (b) For some x, $x^2 = 64$.

** **(a) P(x): x is a whole number** **(c) P(x): x is an irrational number**
(b) P(x): x is a real number **(d) P(x): x is not a rational number.**

5. Write each open sentence in ** as a universal quantified statement, using the symbol $\exists$ or $\forall$.

6. Write each open sentence in ** as an existential quantified statement using the symbol $\exists$ or $\forall$.

7. Rewrite the following quantified statement using the symbols $\exists$ and $\forall$.
   (**Hint**: First, write correctly the P(x) as an open statement.)
   (a) For some triangles, the sum of the measure of the interior angles is 180˚.
   (b) For all triangles, the sum of the measure of the interior angles is 180˚.

8. For each open sentence, assign a value for x that makes the open sentence a true statement.
   (a) $x + 4 = 7$ (c) $x^2 = 16$
   (b) $x - 7 = 4$ (d) $x + 3 = 3 + x$

9. Use a quantifier to make each open sentence in Problem 8 into a true statement.

10. Use a quantifier to make each open sentence in Problem 8 into a false statement.

Open statements, Quantifiers and Quantified Statements

Activity 2

1. The notation $\exists!\, x\, P(x)$ denotes the proposition

   "There exists a unique x such that P(x) is true."

   What are the truth values of the following statements? (where x is an integer)

   (a) $\exists!\, x(x > 1)$
   (b) $\exists!\, x(x^2 = 1)$
   (c) $\exists!\, x(x + 3 = 2x)$
   (d) $\exists!\, x(x = x + 1)$

2. What are the truth values of the following statements? (where x is an integer)
   (a) $\exists!\, x\, P(x) \rightarrow \exists\, x\, P(x)$
   (b) $\forall x\, P(x) \rightarrow \exists!\, x\, P(x)$
   (c) $\exists!\, X \sim P(x) \rightarrow \sim\forall\, x\, P(x)$

3. Write out the quantified statement $\exists!\, x\, P(x)$, where x consists of the integers 1, 2, and 3.

4. Write the negation of each statement without using the expression "It is not true that."
   (a) All athletes over 6 feet tall play basketball.
   (b) Some students work hard at their studies.
   (c) Some professors are not intelligent.
   (d) No man weighs more than 500 pounds.

5. Write each of the expressions in Problem 4 using the symbol $\exists$ and $\forall$.

Theme IV: Logic and Mathematical Reasoning

Lesson 4: Deductive Reasoning

Do You Know?

Valid Arguments

Consider the following two sets of statements.

Argument A:
If Salaam studies, then he will make an A.
If Salaam makes an A, then he will make the Honor Roll.
Salaam studies.

Argument B:
If there is a path connecting each pair of vertices in a graph, then the graph is connected.
In graph G there is a path connecting each pair of vertices.

Can you determine a conclusion that follows from the statements in Argument A? Did you choose the statement "Salaam makes the Honor Roll"? Were you able to deduce a conclusion from Argument B? Did you conclude, "Graph G is connected"? If so, then you are reasoning clearly. Further, you have demonstrated a skill at two of the classic patterns of reasoning that form the foundation of deductive reasoning.

Consider the collection of statements

If Salaam studies, then he will make an A.
If Salaam makes an A, then he will make the Honor Roll.
Salaam studies.
****
Therefore, Salaam makes the Honor Roll.

That is an example of a **deductive argument**. Both in mathematics and in everyday affairs, arguments arise in which we need to deduce a correct conclusion from a given set of statements. In general, an argument consists of two parts: a set of two or more statements called **premises** and a single statement called the **conclusion**. An argument is **valid** if the conclusion is true in every circumstance in which the conjunction of the premises is true. If, in some case, the conjunction of the premises is true and the conclusion is false, then the argument is **invalid**. Invalid arguments are sometimes called **fallacies**. Note that a valid argument may have a conclusion that is false if any one of the premises fails to be true.

Deductive Reasoning

Arguments are often written symbolically by naming the statements that form the argument:

If inflation occurs, then the price of cars increases. $p \rightarrow q$
Inflation occurs. p
****
Therefore, the price of automobiles increases. $\therefore q$

Three dots $\therefore$ are read as **therefore**.

When verbal arguments are converted into symbolic form, we find that the same patterns often occur. For example, the pattern in the previous paragraph occurs frequently.

$(p \rightarrow q) \wedge p;\ \therefore q$ or $[(p \rightarrow q] \wedge p] \rightarrow q$ is a tautology.

The crucial fact about a valid argument is that the truth of its conclusion follows necessarily or logically from the truth of its premises. It is impossible to have a valid argument with true premises and a false conclusion. When an argument is valid and its premises are true, the truth of the conclusion is said to be inferred or deduced from the truth of the premises. If a conclusion is not true, then it is not a valid deduction.

There are basically four (4) rules of inference used in statements involving quantifiers where valid arguments are inferred, deduced, or constructed. These rules of inference are used extensively in mathematical arguments and deductive reasoning, often without being explicitly mentioned.

Universal instantiation is the rule of inference used to conclude that P(c) is true, where c is a particular member of the universe of discourse, given the premise $\forall xP(x)$. Universal instantiation is used when we conclude from the statement "All women are wise" that 'Lisa is wise," where Lisa is a member of the universe of discourse of all women.

Universal generalization is the rule of inference which states that $\forall xP(x)$ is true, given the premise that P(c) is true for all elements c in the universe of discourse. Universal generalization is used when we show that $\forall xP(x)$ is true by taking an arbitrary element c from the universe of discourse and showing that P(c) is true. The element c that we select must be an arbitrary, and not specific, element of the universe of discourse. Universal generalization is used implicitly in many proofs in mathematics and is seldom mentioned explicitly.

Existential instantiation is the rule which allows us to conclude that there is an element c in the universe of discourse for which P(c) is true if we know that $\exists\ xP(x)$ is true. We cannot select an arbitrary value of c here, but rather it must be a c for which P(c) is true. Usually we have no knowledge of what c is, only that it exists. Since it exists, we may give it a name (c) and continue our argument.

Deductive Reasoning

Existential generalization is the rule of inference which is used to conclude that ∃ *xP(x)* is true when a particular element *c* with P(c*)* true is known. That is, if we know one element *c* in the universe of discourse for which P(*c*) is true, then we know that ∃ *xP(x)* is true.

We summarize these rules of inference in Table A.

TABLE A **Rules of Inference for Quantified Statements.** ***U* is the Universe of Discourse, where *x* belongs.**	
Rule of Inference	***Name***
$\dfrac{\forall x P(x)}{\therefore P(c) c \in U}$	**Universal instantiation**
$\dfrac{P(c)\mathbf{c} \in U}{\therefore \wedge x P(x)}$	**Universal generalization**
$\dfrac{\exists x P(x)}{\therefore P(c)\mathbf{c} \in \mathbf{U}}$	**Existential instantiation**
$\dfrac{P(c)\mathbf{c} \in \mathbf{U}}{\therefore \exists x P(x)}$	**Existential generalization**

One of the fundamental uses of deductive reasoning in mathematics is that of proving theorems.

These rules of inference are used in mathematics primarily to prove propositions. Different methods of proofs are used in the proving of theorems.

Methods of proof

Two important questions that arise in the study of mathematics are: (1) When is a mathematical argument correct? (2) What methods can be used to construct mathematical arguments? This section helps answer these questions by describing various forms of correct and incorrect mathematical arguments.

A **theorem** is a statement that can be shown to be true. (Theorems are sometimes called propositions, facts, or results.) We demonstrate that a theorem is true with a sequence of statements that form an argument, called a **proof**. To construct proofs, methods are needed to derive new statements from old ones. The statements used in a proof can include axioms or postulates, which are underlying assumptions about mathematical structures, the hypotheses of the theorem to be proved, and previously proved theorems. The **rules of inference**, which are the means used to draw conclusions from other assertions, tie together the steps of a proof.

Deductive Reasoning

The terms lemma and corollary are used in the proof of other theorems. A **lemma** (plural **lemmas** or **lemmata**) is a simple theorem used in the proof of other theorems. Complicated proofs are usually easier to understand when they are proved using a series of lemmas, where each lemma is proved individually. A **corollary** is a proposition that can be established directly from a theorem that has been proved. A **conjecture** is a statement whose truth value is unknown. When a proof of conjecture is found, the conjecture becomes a theorem. Many times conjectures are shown to be false, so they are not always theorems.

The methods of proof discussed in this lesson are important not only because they are used to prove mathematical theorems, but also for their many applications of deductive reasoning in everyday challenges.

We end this lesson by naming a number of different kinds of proofs and giving examples of these methods of proofs. Again, proving theorems is the most important use of deductive reasoning in mathematics.

Direct proofs

The implication $p \rightarrow q$ can be proved by showing that if p is true, then q must also be true. This shows that the combination p true and q false never occurs. A proof of this kind is called a **direct proof**. To carry out such a proof, assume that p is true and use the rules of inference and theorems already proved to show that q must also be true.

Example 1

Theorem "If n is an odd integer, then n^2 is an odd integer." (Give a direct proof.)

Solution

Assume that the hypothesis of this implication is true, namely, suppose that n is odd. Then $n = 2k + 1$, where k is an integer. It follows that $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(k^2 + 2k) + 1$. Therefore, n^2 is an odd integer (it is one more than twice an integer).

Indirect proofs

Since the implication $p \rightarrow q$ is equivalent to its contrapositive, $\sim q \rightarrow \sim p$, the implication $p \rightarrow q$ can be proved by showing that its contrapostive, $\sim q \rightarrow \sim p$, is true. This related implication is usually proved directly, but any proof technique can be used. An argument of this type is called an **indirect proof.**

Deductive Reasoning

Example 2

Theorem "If $3n + 2$ is odd, then n is odd." (Give an indirect proof.)

Solution

Assume that the conclusion of this implication is false; namely, assume that n is even. Then $n = 2k$ for some integer k. It follows that $3n + 2 = 3(2k) + 2 = 6k + 2 = 2(3k + 1)$, so $3n + 2$ is even (since it is a multiple of 2) and therefore is not odd. Because the negation of the conclusion of the implication implies that the hypothesis in false, the original implication is true.

Proofs by contradiction

Proofs by contradiction is another approach that we can use when neither a direct nor an indirect proof succeeds.

Suppose that a contradiction q can be found so that $\sim p \rightarrow q$ is true, that is $\sim p \rightarrow \mathbf{F}$ is true. Then the proposition $\sim p$ must be false. Consequently, p must be true. This technique can be used when a contradiction, such as $r \wedge \sim r$, can be found so that it is possible to show that the implication $\sim p \rightarrow (r \wedge \sim r)$ is true. An argument of this type is called **proof by contradiction**.

Example 3

Theorem "If $3n + 2$ is odd, then n is odd."

Solution

We assume that $3n + 2$ is odd and that n is not odd, so that n is even. Following similar steps as in the indirect proof, we can show that if n is even, then $3n + 2$ is even. This contradicts the assumption that $3n + 2$ is odd, completing the proof.

There are several other methods of proofs, such as

- **Proof by cases** - a proof of an implication where the hypothesis is a disjunction of propositions that shows that each hypothesis separately implies the conclusion;
- **Vacuous proof –** a proof that the implication $p \rightarrow q$ is true based on the fact that p is false
- **Trivial proof –** a proof that the implication $p \rightarrow q$ is true based on the fact that q is true

and others. However, the three (3) that we have given in this lesson, with examples, are the most frequently used.

Deductive Reasoning

Activity 1

Part I: True or False (Explain your answer.)

____ 1. Every proposition is a theorem.
____ 2. Every theorem is a proposition.
____ 3. Every theorem is a lemma.
____ 4. Every lemma is a theorem.
____ 5. Every contingency is always false.
____ 6. All the truth values of a tautology are false.
____ 7. All the truth values of a contradiction are true.
____ 8. A fallacy is a method of proof.
____ 9. A counterexample is a method of proof.
____ 10. There are only three ways to prove a theorem (1) direct, (2) indirect, (3) by contradiction.

Part II: Formulate the arguments of Exercises 1 – 5, symbolically, and determine whether each is valid. Let

p: I study hard. *Q: I get A's* *r: I get rich*

1. If I study hard, then I get A's.
   I studied hard.
   ∴ I got A's.

2. If I study hard, then I get A's.
   If I do not get rich, then I do not get A's.
   ∴ I got rich.

3. I study hard if and only if I get rich.
   I got rich.
   ∴ I studied hard.

4. If I study hard or I get rich, then I get A's.
   I got A's.
   ∴ If I do not study hard, then I get rich.

5. If I study hard, then I get A's or I get rich.
   I do not get A's and I do not get rich.
   ∴ I did not study hard.

Deductive Reasoning

Activity 2

Prove these theorems by direct proof, indirect proof, or by contradiction.

Theorem 1 If x and y are even integers, then x + y is even.

Theorem 2 If x is an odd integer and y is an even integer, then the product *xy* is an even integer.

Theorem 3 If x is an odd integer and y is an odd integer, then x + y is an even integer.

Theorem 4 If x is an even integer an y is an odd integer, then x + y is an odd integer.

Theorem 5 For an integer n, n^2 is even, then n is even.

1. Prove theorem 1.

2. Prove theorem 2.

3. Prove theorem 3.

4. Prove theorem 4.

5. Prove theorem 5.

6. – 10.
   Create your own problems or theorems that you would like to solve.

Theme IV: Logic and Mathematical Reasoning

Lesson 5: Inductive Reasoning

Do You Know?

Inductive reasoning is essentially the opposite of deductive reasoning. It involves trying to create general principles by starting with many specific instances. For example, in inductive geometry you might measure the interior angles of a group of randomly drawn triangles. When you discover that the sum of three angles is 180˚ regardless of the triangle, you would be tempted to make a generalization about the sum of the interior angles of a triangle. Bringing forward all these separate facts provides evidence in order to help support your general statement about the interior angles.

This is the kind of reasoning used if you have gradually built up an understanding of how something works. Rather than starting with laws and principles and making deductions, most people collect relevant experience and try to construct principles from it.

Inductive reasoning progresses from observations of individual cases to the development of a generality. Here are some general examples of inductive reasoning:

Example 1

A person drives down a particular road at rush hour several times and finds the traffic terrible each time. Therefore, this is a good road to avoid at rush hour.

Example 2

Well, I've observed many patients receive a certain drug combination, and there never have been observed any problems with it. Therefore, this drug combination seems not to have any negative side effects.

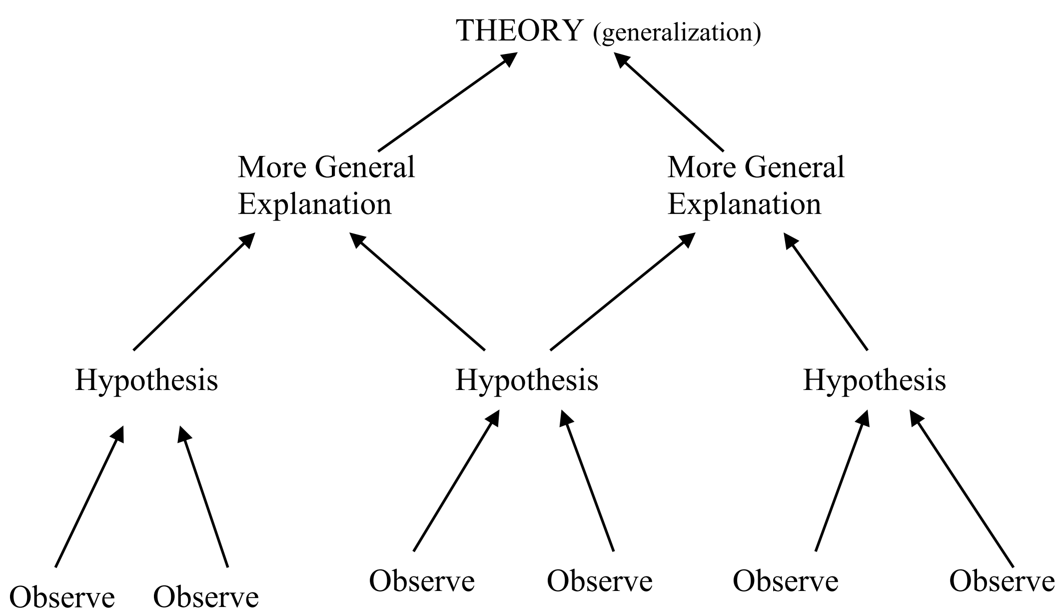
Inductive arguments are always open to question as, by definition, the conclusion is bigger than the evidence on which it is based. **Inductive reasoning** is the process of arriving at a conclusion based on a set of observations. In itself, it is not a valid method of proof. Just because a person observes a number of situations in which a pattern exists does not mean that the pattern is true for all situations.

Inductive Reasoning

For example, inductive reasoning is often used in geometry and other areas of mathematics. One might observe that in a few given rectangles, the diagonals are congruent (same size). The observer could inductively reason that in all rectangles, the diagonals are congruent. Although we know this to be generally true, the observer has not proved it through his limited observations. However, he could prove his hypothesis using other means and come out with a theorem (a proven statement). In this case, as in many others, inductive reasoning led to a suspicion, or more specifically, a hypothesis, that ended up being true. However, in all cases the hypothesis is not true.

The power of inductive reasoning, then, does not lie in its ability to prove mathematical statements. In fact, inductive reasoning can never be used to provide proofs. Instead, inductive reasoning is valuable because it allows us to form ideas about groups of things in real life. In geometry, inductive reasoning helps us organize what we observe into succinct geometric hypotheses that we can prove using other, more reliable methods. Whether we know it or not, the process of inductive reasoning almost always is the way we form ideas about things. Once those ideas form, we can systematically determine (using formal proofs) whether our initial ideas were right, wrong, or somewhere in between.

INDUCTIVE REASONING IS A BOTTOM UP PROCESS



Begins with Observations
BOTTOM UP PROCESS

Inductive Reasoning

Although inductive reasoning is not a method of proof, there are two ideas associated with inductive reasoning that are important to mathematics and other aspects of reality.

A. Inductively Defined Sets (a construction technique),

B. The Principal of Mathematical Induction (which is the foundation of proof by Mathematical Induction).

These two ideas help solve real problems. There are usually two (2) parts to solving a problem. The first part is to make a guess at what one believes might be a solution. The second part is to verify that this guess is or is not correct. We will now show how these ideas solve problems; especially mathematically.

Inductively Defined Sets

When we write down an informal statement such as A = {3, 5, 7, 9, …}, most of us will agree that we mean the set $A = \{2k + 3 \mid k \in \mathbf{N}$. Another way to describe A is to observe that $3 \in A$, that $x \in A$ implies $x + 2 \in A$, and that the only way an element gets in A is by these two steps. This description of A has three ingredients:

1. There is a starting element (3).

2. There is a construction operation to build new elements from existing elements (addition by 2).

3. There is a statement that no other elements are in set.

Some Inductive Definitions

This process in an example of an inductive definition of a set. The set of objects defined is called an inductive set. An **inductive set** consists of objects that are constructed, in some way, from objects that are already in the set. So nothing can be constructed unless there is at least one object in the set to start the process. Inductive sets ate important in computer science because the objects can be used to represent information and the construction rules can often be programmed. A formal definition is as follows:

An inductive definition of a set S consists of three (3) steps:

Basis:	Specify one or more elements of S.
Induction:	Give one or more rules to construct new elements of S form existing elements of S.
Closure:	State that S consists exactly of the elements obtained by the basis and induction steps. This step is usually assumed rather than stated explicitly.

The closure step is a very important part of the definition. Without it, there could be lots of sets satisfying the first two steps of an inductive definition.

Inductive Reasoning

Example 3

The natural number **N** can be defined as an inductive set.

The set of natural numbers N = {1, 2, 3…} is an inductive set. Its basis element is 0, and we can construct a new element from an existing one by adding the number 1. So we can write an inductive definition for N in the following way.

Basis: $1 \in \mathbf{N}$.
Induction: If $n \in \mathbf{N}$, then $n + 1 \in \mathbf{N}$.

The constructors of **N** are the integer 1 and the operation that adds 1 to an element of **N**. The operation of adding 1 to n is called the successor function, which we write as

$$\text{succ}(n) = n + 1.$$

Using the successor function, we can rewrite the induction step in the above definition of N in the alternative form

If $n \in \mathbf{N}$, then $\text{succ}(n) \in \mathbf{N}$.

So we can say that **N** is an inductive set with two constructors, 1 and succ.

Example 4

Some familiar odd numbers: A = {1, 3, 7, 15, 31, …}, can be defined as an inductive set. An inductive definition of A can be written as follows:

Basis: $1 \in A$.

Induction: If $x \in A$, then $2x + 1 \in A$.

Example 5

Fibonacci numbers can be defined as an inductive set, recursively, as follows:

fib(0) = 0,
fib(1) = 1,
fib(n) = fib(n – 2) + fib(n – 1) if n > 1.

The Principle of Mathematical Induction

Let P(n) be a statement involving natural numbers. To prove that P(n) is true for all integers $n \geq m$ (for $m \in \mathbf{Z}$), perform the following two (2) steps:

1. Prove that P(m) is true.

2. Assume that P(k) is true for an arbitrary $k \geq m$. Then prove that P(k + 1) is true.

The principle of mathematical induction is a technique to prove that infinitely many statements are true in just two steps. Quite a savings in time. Let's look at an example. This proof technique is just what one needs to prove examples like the following.

Inductive Reasoning

Example 6

Prove by mathematical induction Let P(n): $1 + 3 + 5 + \ldots + (2n - 1) = n^2$

Solution

Consider P(n): $1 + 3 + 5 + \ldots + (2n - 1) = n^2$

We must first complete the basis step; that is, we must show that P(1) is true. Then we must carry out the inductive step; that is, we must show the P(k + 1) is true when P(k) is assumed to be true.

BASIS STEP: P(1): $1 = 1^2 = 1$; thus, P(1) is true.
INDUCTIVE STEP: To complete the inductive step we must show that the proposition $P(k) \rightarrow P(k + 1)$ is true for every positive integer k. To do this, assume that P(k): $1 + 3 + 5 + \ldots + (2k - 1) = k^2$ is true for a positive integer k;

Now it must be shown that P(k + 1) is true, assuming that P(k) is true. We know that P(k + 1) has this expression $1 + 3 + 5 + \ldots + (2k - 1) + (2k + 1) = (k + 1)^2$
Assuming that P(k) is true, it follows that

$$
\begin{aligned}
P(k + 1): 1 + 3 + 5 + \ldots + (2k - 1) + (2k + 1) &= k^2 + (2k + 1) \\
&= k^2 + 2k + 1 \\
&= (k + 1)(k + 1) \\
&= (k + 1)^2.
\end{aligned}
$$

This shows that P(k + 1) follows from P(k). This concludes our proof by mathematical induction. We now use the Principle of Mathematical Induction to prove an inequality.

Example 7

Use mathematical induction to prove the inequality

$$n < 2^n$$

for all positive integers n.

Solution

Let P(n) be the proposition "$n < 2^n$."

BASIS STEP: P(1) is true, since $1 < 2^1 = 2$.
INDUCTIVE STEP: Assume that P(k) is true for the positive integer k. That is, assume that $k < 2^k$. We need to show that P(k + 1) is true. That is, we need to so that $k + 1 < 2^{k+1}$. Adding 1 to both sides of $k < 2^k$, and then nothing that $1 \leq 2^k$, gives

$$k + 1 < 2^k + 1 \leq 2^k + 2^k = 2^{k+1}.$$

We have shown that P(k + 1) is true, namely, that $k + 1 < 2^{k+1}$, based on the assumption that P(k) is true. The induction step is complete.

Therefore, by the principle of mathematical induction, it has been shown that $n < 2^n$ is true for all positive integers *n*.

Inductive Reasoning

Activity 1

*** Define each of the following sets inductively.**

1. S1 = {1, 3, 5, 7, …}
2. S2 = {0, 2, 4, 6, 8, …}
3. S3 = {-3, -1, 1, 3, 5, …}
4. S4 = {…, -7, -4, -1, 2, 5, 8 , …}
5. S5 = {1, 4, 9, 16, 25, …}
6. S6 = {1, 3, 7, 15, 31, 63, ….}
7. S7 = $A \cup B$ = {2, 4, 8, 16, …} $\cup$ {3, 7, 11, 15, …}
8. Construct recursively f(n) = {0, 1, …, n)
9. Construct recursively f(n) = {0, 4, 8, 16, …, n}

Given f(0) = 0, f(1) = 1, f(n) = (n – 2) + (n – 1) for n > 1

10. List f(1), f(2), …, f(12)

Inductive Reasoning

Activity 2

1. Define the set S = {4, 7, 10, 13, …} ∪ {6, 9, 12, …} inductively. Please not that S = S ∪ S which means that two sets have to be defined inductively: S and S

2. Construct recursively f(n, k) = {0, k, 2k, 3k, …, nk}

3. Construct recursively f(n, k) = {n, n + 1, n + 2, …, n + k}

4. Prove by mathematical induction that

   P(n): 2 + 6 + 10 + …+ (4n – 2) = $2n^2$

5. Show that n! < n^n for n > 1

   (Use the Principle of Mathematical Induction.)

6. \- 10.

   Create five (5) problems that you wish to solve involving inductive sets or proofs by the Principle of Mathematical Induction.

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